Quantum Stochastic Convolution Cocycles

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Ocho y ocho son dieciséis y el que cuenta.
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Abstract

A concept of quantum stochastic convolution cocycle is introduced and studied in two different contexts – purely algebraic and operator space theoretic. A quantum stochastic convolution cocycle is a quantum stochastic process k on a coalgebra $\mathcal{A}$ satisfying the convolution cocycle relation

$$k_{s+t} = (k_s \otimes (\sigma_s \circ k_t)) \circ \Delta$$

and the initial condition $k_0 = \iota \circ \epsilon$, where Δ and ϵ denote the coproduct and counit of $\mathcal{A}$, σ denotes the time shift on operators acting on the symmetric Fock space over $\mathbb{R}_+$ and ι is the unital embedding of $\mathbb{C}$ in the algebra of bounded operators on the Fock space. The notion generalises that of quantum Lévy process, which in turn is a noncommutative probability counterpart of classical Lévy process on a group.

Convolution cocycles arise as solutions of quantum stochastic differential equations. In turn every sufficiently regular cocycle satisfies an equation of that type. This is proved along with the corresponding existence and uniqueness of solutions for coalgebraic quantum stochastic differential equations. The stochastic generators of unital *-homomorphic cocycles are characterised in terms of structure maps on a *-bialgebra. This yields a simple proof of the Schürmann Reconstruction Theorem for a quantum Lévy process; it also yields a topological version for a quantum Lévy process on a C^*-bialgebra. Precise characterisation of the stochastic generators of completely positive and contractive quantum stochastic convolution cocycles in the C^*-algebraic context is given, leading to some dilation results. A few examples are presented and some interpretations offered for quantum stochastic convolution cocycles and their stochastic generators on different types of *-bialgebra.

The techniques used for the analysis of convolution cocycles in the purely algebraic and operator space theoretic context are distinct. In the first case the basic tool is the Fundamental Theorem on Coalgebras. In the second the key method is the application of the so-called R-transformation linearising quantum convolution.

Contents

Chapter 1

Introduction

This thesis is concerned with the investigation of quantum stochastic convolution cocycles in both a purely algebraic context and in the context of compact quantum groups. The notion has been introduced by J.M. Lindsay and the author in [LS_1]. Quantum stochastic cocycles are quantum stochastic processes living on a (symmetric) Fock space of a Hilbert space of vector-valued square-integrable functions on the positive half-line and satisfying a cocycle relation with respect to the Fock space shift. This notion derives from the idea of the cocycle structure that may be associated to continuous-time classical Markov processes (the precise analogy and connections are exploited in [LiS]; the example of Brownian motion cocycles cast in the language used in this thesis may be found in [Lin]). Quantum stochastic cocycles are basic objects of interest in quantum stochastic analysis ([Par], [Mey], [Bia], [Lin]) and in the more general theory of noncommutative white noise ([HKK]). They were first introduced in the noncommutative probability literature in the late 1970s by L. Accardi and his collaborators ([Acc], [AFL]). Later developments unearthed connections to other areas of mathematics. As example we can mention the application of the analysis of quantum stochastic cocycles to resolve certain open questions in Arveson's theory of product systems ([Arv]). On the other hand there is a well developed theory of quantum Lévy processes, originating in the paper [ASW] of L. Accardi, M. Schürmann and W. von Waldenfels and further extensively developed especially by the second mentioned author (see [Fra], [Sch] and references therein). Quantum Lévy processes generalise classical stationary, independent increment stochastic processes on groups. Close examination of the two areas described above has naturally led to the idea of quantum stochastic convolution cocycle.

As usual in noncommutative mathematics, in order to 'quantise' one focuses on the appropriate class of functions on the underlying set of the classical structure. When the interest lies in topological aspects of a space, one starts with the algebra of continuous functions, leading to the theory of C^*-algebras, and further, via introducing differential structure and the K-theory to the noncommutative geometry of A. Connes. When the underlying set is considered as a

measure space, the relevant object is the algebra of all measurable functions, leading to the land of von Neumann algebras. By analogy we see that quantum stochastic convolution cocycles should 'act on' a generalisation of the algebra of complex-valued functions on a group or, to be more precise, semigroup with identity, namely on a *-bialgebra. These, depending on whether the topology is taken into account, may be considered in two different categories, purely algebraic and operator-space theoretic. This dichotomy influences the structure of the thesis, which is described in the end of this chapter.

Classical theory of probability on algebraic structures

This thesis is aimed to be a contribution to the theory of noncommutative probability on quantum/algebraic structures. In this subsection we indicate the amazing variety and richness of the classical theory. This is done first by presenting major ideas and concepts of extending the standard probability theory, concerned with real-valued random variables, to more general algebraic structures. The choice is clearly subjective, and is not intended to represent neither fair summary nor general introduction to the subject. For this we refer to a beautiful early book [Gre] and to the impressive volume [He2] (see also [He1], [He3], [BlH]). In the second part we concentrate on a particular case of probability measures on compact groups, which is of importance for the following chapters.

Let us start with the observation that the afore-mentioned extensions of standard probability theory may be seen at least on two basic levels. We may change the target space of the random variables in question and consider random variables taking values in infinite dimensional vector spaces, in abelian or nonabelian groups, or in (Banach) algebras. We may also consider typical distributions on the real line or complex plane, combine them using algebraic operations, and then ask questions concerning the objects obtained, as is done for example in the theory of random matrices.

The importance of investigating properties and looking for answers to new problems arising in the constructions described above stems from multiple

sources. Firstly, there exist concrete physical models that can be usefully described in the algebraic language. To mention the simplest examples, behaviour of a ball with a fixed centre floating in a randomly moving fluid may be analysed via the theory of random variables with values in the rotation group SO(3); a chain of devices transforming the initial data according to certain linear prescription and subjected to errors may be viewed as a dynamical system whose evolution is governed by consecutively applied random matrices. Secondly, a probabilistic approach may be helpful in answering deterministic questions, an idea which has been successfully pursued by the Polish school of functional analysis (e.g.by constructing Banach spaces with desired properties as suitable limits of randomly generated finite-dimensional spaces). Thirdly, attempts to generalise allow us to understand better the particular case of probability theory on $\mathbb{R}$, both its advantages and limitations. Finally there is the most important reason according to the author, namely pure intellectual curiosity.

Extensions of classical results usually present various technical problems. The main tools of the standard theory, such as Fourier analysis, need to be reformulated and in the process often lose at least some of their well-known properties. Many theorems need additional assumptions, automatically satisfied by real- or complex-valued random variables. In most cases however, functional analytic techniques remain useful; their full strength is often revealed only in such a general context.

Let us now illustrate somewhat vague statements above, and provide a bridge with the considerations in further chapters. Assume that G is a compact semigroup. Its multiplicative structure provides the convolution action on $M(G)$, the Banach space of all regular complex-valued Borel measures on G with total variation norm:

$$\mu \star \nu(f) = \int_G \int_G f(st) d\mu(s) d\nu(t), \quad \mu, \nu \in M(G),\ f \in C(G),$$

where the isometric isomorphism $M(G) \cong C(G)^*$ is used ($C(G)$ denotes the Banach space of all continuous functions on G with the supremum norm). It

is easy to see that $(M(G), \star)$ becomes a Banach algebra. Further if G has a neutral element e, the measure δ_e (the Dirac measure concentrated in e) is the unit of $(M(G), \star)$. This allows us to introduce the following notion: a family $\{\mu_t : t \geq 0\}$ of measures in $M(G)$ is called a *convolution semigroup of measures* if

$$\mu_{s+t} = \mu_s \star \mu_t, \quad s, t \geq 0, \qquad \text{and} \qquad \mu_0 = \delta_e.$$

It is called *weakly continuous* (or more correctly weak*-continuous) if $|\mu_t(f) - \mu_0(f)| \xrightarrow{t \to 0^+} 0$ for each $f \in C(G)$, it is called *norm continuous* if $\|\mu_t - \mu_0\| \xrightarrow{t \to 0^+} 0$.

Already these, apparently simple notions allow interesting, and in general highly nontrivial, questions to be asked. For example,

(a) how do (norm continuous, weakly continuous) convolution semigroups of measures look like? Can they be classified?

(b) given a measure $\mu \in M(G)$, when does it embed in a convolution semigroup of measures (that is, when does the convolution semigroup of measures $\{\mu_t : t \geq 0\}$ exist such that $\mu = \mu_1$)? If it does embed, is the embedding unique?

(c) what is the limit behaviour of the semigroup? Given a measure $\mu \in M(G)$, when do the sequences $(\mu^{\star k})_{k=1}^{\infty}$, $(\frac{1}{k} \sum_{n=1}^{k} \mu^{\star k})_{k=1}^{\infty}$ converge? What are the possible limits?

The typical method used for approaching these problems is to transform questions concerning convolution semigroups to more familiar questions concerning certain semigroups of operators. Define for each $\mu \in M(G)$ the map $P_\mu : C(G) \to C(G)$ by

$$P_\mu(f)(s) = \int_G f(st) d\mu(t),$$

$f \in C(G)$, $s \in G$. When μ is a probability measure, the reader will have recognised here the so-called Markov operator. Not surprisingly, the problems

above become easier when attention is restricted to norm-continuous semigroups. These can be shown to correspond precisely to distributions of compound Poisson processes ([Gre]). In full generality, the answers to (a) and (b) depend on additional properties of the (semi)group G (one should mention for example the celebrated Hunt formula from [Hun], characterising generators of weakly continuous convolution semigroups of measures on Lie groups - see [He2]). Here, as in the case of $\mathbb{R}$, the Fourier transform, this time understood in the language of the Peter-Weyl theory of unitary representations of compact groups, is an indispensable tool. The limits of sequences in (c), when they exist, are clearly idempotent measures (that is measures satisfying $\mu \star \mu = \mu$), and one can show that if G is a group, the idempotent probability measures correspond to Haar measures on subgroups. The corresponding property does not hold in the quantum case (for more information on that see [Pal]).

The above well-known facts are quoted here, as the same methods will be echoed in Chapter 3, and especially in Chapter 4, in a noncommutative context. There we will approach the stochastic process by looking at the convolution semigroups of functionals given by its distribution, and then proceed with the idea of considering the corresponding semigroups of operators (formalised in the guise of the R-map introduced in Section 4.1). Further comments on relations between the topic of this subsection and the main subject of this thesis are given in Section 4.8.

Summary

Here we present a summary of the main results of the thesis, expanding on the abstract.

A concept of quantum stochastic convolution cocycle is introduced and studied in two different contexts, namely purely algebraic and operator space theoretic. A quantum stochastic convolution cocycle is a quantum stochastic process k on a coalgebra $\mathcal{A}$ (with coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and counit

$\epsilon : \mathcal{A} \to \mathbb{C}$) satisfying the convolution cocycle relation

$$k_{s+t} = (k_s \otimes (\sigma_s \circ k_t)) \circ \Delta$$

and the initial condition

$$k_0 = \iota_{\mathcal{F}} \circ \epsilon.$$

The functional equation is for operators on a symmetric Fock space $\mathcal{F}$ over a Hilbert space of square-integrable vector-valued functions on $\mathbb{R}_+$. Here σ denotes the time shift on the Fock space operators and $\iota_{\mathcal{F}}$ is the unital embedding of $\mathbb{C}$ in the algebra of bounded operators on $\mathcal{F}$. The notion generalises the notion of quantum Lévy process, which in turn is a noncommutative probability counterpart of classical Lévy process on a group. Moreover it is a convolution counterpart to a previously studied structure, referred to here as *standard* quantum stochastic cocycle. Such a cocycle is a quantum stochastic process, again consisting of Fock space operators, satisfying the cocycle relation $k_{s+t} = \widehat{k}_s \circ \sigma_s \circ k_t$ (where $\widehat{k}_s$ denotes a certain natural extension of k_s) and the initial condition $k_0 = \iota_{\mathcal{F}}$.

Convolution cocycles arise as solutions of certain kinds of linear constant-coefficient quantum stochastic differential equations, as do standard cocycles. We show, under suitable assumptions on the coefficients, existence and uniqueness for solutions of coalgebraic quantum stochastic differential equations with initial condition given by the counit; these turn out to be quantum stochastic convolution cocycles. Conversely, we show that every sufficiently regular convolution cocycle satisfies an equation of that type. The stochastic generators of unital *-homomorphic cocycles on a *-bialgebra may be characterised in terms of structure maps, equivalently by Schürmann triples. This yields a simple proof of the Schürmann Reconstruction Theorem for a quantum Lévy process; it also yields a topological version for quantum Lévy process on a C^*-bialgebra. Precise characterisation of the stochastic generators of completely positive and contractive quantum stochastic convolution cocycles in the C^*-algebraic context is given, leading to two theorems on dilations of such cocycles

to *-homomorphic convolution cocycles. Finally we present a few examples and offer some interpretations of quantum stochastic convolution cocycles and their stochastic generators on different types of *-bialgebra.

The techniques used for the analysis of cocycles in the purely algebraic and operator space theoretic context are distinct. In the first case the basic tool is the Fundamental Theorem on Coalgebras, as exploited by Schürmann in his pioneering work. In the second the key method is the application of the so-called R-transformation linearising quantum convolution to the well-established theory of standard quantum stochastic cocycles.

Description of the contents

Having briefly described classical inspirations and summarised the main results, the rest of this chapter consists of a detailed description of the contents of the thesis. Quantum stochastic is abbreviated to QS in the sequel. Chapter 2 contains preliminaries. After introducing in Section 2.1 the general notations, we proceed in Section 2.2 to recall the basic notions of operator space theory and the definition of matrix spaces. Section 2.3 contains standard facts concerning Fock spaces and defines QS processes. The last part of Chapter 2, Section 2.4, introduces the QS integration of R.L. Hudson and K.R. Parthasarathy. Fundamental Formulae and Fundamental Estimate are quoted, and iterated QS integrals described; the notation used is mainly modelled on the lecture notes [Lin].

In Chapter 3 QS convolution cocycles are investigated in the purely algebraic context. The chapter starts by introducing coalgebras and convolution structure provided by the coproduct in Section 3.1 and proceeds with the definition of QS convolution cocycles and their associated semigroups in Section 3.2. Section 3.3 contains the existence and uniqueness results for solutions of coalgebraic QS differential equations; it also describes basic properties of the solutions, including a form of Hölder continuity. The fact that the solutions of coalgebraic QS differential equations are QS convolution cocycles is proved in Section 3.4; there also a converse is obtained - all Hölder continuous QS convo-

lution cocycles must arise in this way. Section 3.5 is devoted to multiplicative properties of the cocycles on *-bialgebras. As the operators in question may be (and usually are) unbounded, the analysis concentrates rather on properties of scalar products than the actual composition. The Itô Formula for iterated QS integrals is expressed via so-called matrix-sum kernels, following [LW_4], and the general structure of the stochastic generators of weakly multiplicative cocycles is obtained. The known characterisation of generators of *-homomorphic QS convolution cocycles in terms of Schürmann triples follows, and in Section 3.6 we recall the definition of quantum Lévy processes due to L. Accardi, M. Schürmann and W. von Waldenfels ([ASW]) and give an alternative proof of the Schürmann Reconstruction Theorem ([Sch]). Section 3.7 is devoted to certain perturbation formulas (see also [Fra]). The last section of Chapter 3 contains a definition of opposite QS convolution cocycles and indicates corresponding versions of the results of previous sections.

In Chapter 4 we continue the investigation of QS convolution cocycles, this time in the operator space theoretic (or C^*-algebraic) context. It begins with Section 4.1 introducing operator space coalgebras and convolution semigroups of functionals. The stress is put on the so-called R-map which transforms convolution semigroups into semigroups of operators, whilst respecting their relevant continuity properties. In Section 4.2 QS convolution cocycles on operator space coalgebras are defined. This time it is necessary to introduce also the notion of a weak QS convolution cocycle. Later the basic facts on standard QS cocycles are recalled (following [LW_1]), and the R-map is shown to yield a transformation between the two classes of cocycles. Section 4.3 contains the existence and uniqueness results for solutions of QS differential equations in the context of operator spaces, with completely bounded coefficients and nontrivial initial conditions. The methods extend those of [LW_3]. Coalgebraic QS differential equations are the topic of Section 4.4. They may be viewed as a particular case of the equations described in the previous section. Their solutions are weak QS convolution cocycles. From the corresponding fact in the theory of standard QS cocycles and properties of the R-map, in Section 4.5

we prove that every Markov-regular completely positive and contractive QS convolution cocycle on a C^*-hyperbialgebra satisfies a coalgebraic QS differential equation. The precise structure of stochastic generators of such cocycles is also described. Section 4.6 is devoted to *-homomorphic, or more generally weakly multiplicative, cocycles on C^*-bialgebras. Their stochastic generators may be again, as in Chapter 3, expressed in terms of Schürmann triples. Two possible notions of quantum Lévy process on a C^*-bialgebra are proposed and a version of the Schürmann Reconstruction theorem for such processes is given. The question of dilating completely positive QS convolution cocycles on C^*-bialgebras to *-homomorphic ones is addressed in Section 4.7. Two possible forms of dilations are described, corresponding to the results obtained for standard QS cocycles respectively in [GLSW] and in [GLW]. The proofs exploit the characterisation of the stochastic generators obtained earlier. Section 4.8 contains several examples and comments on QS convolution cocycles on specific types of OS coalgebras. Unital *-homomorphic cocycles are discussed in three main cases: commutative (algebras of continuous functions on compact groups), cocommutative (universal C^*-algebras of discrete groups) and genuinely quantum (where the most satisfactory results are obtained for full compact quantum groups). The last part of this section is devoted to recalling the standard conditional expectation construction of C^*-hyperbialgebras, and describing the QS convolution cocycles on the structures so-obtained. Finally the last short section discusses the notion of QS convolution cocycles on multiplier C^*-bialgebras (locally compact quantum semigroups) and poses some of the problems associated with the developing of theory for these.

There are two appendices. In Appendix A the automatic continuity properties of (π_1, π_2)-derivations (for representations π_1, π_2 of a C^*-algebra) are discussed, based on the celebrated work of S. Sakai, J.R. Ringrose and E. Christensen on (standard) derivations on C^*-algebras. These results are needed for the description of the stochastic generators of *-homomorphic cocycles in Section 4.6. Appendix B recalls basics of the theory of Hopf *-algebras and their corepresentations ([KlS]), useful for understanding the examples in

Section 4.8.

Most of the contents of Chapter 3 have been published in [LS_1] and the presentation here is patterned on this paper, with certain transpositions of the material, so that the notion of convolution cocycles is introduced at the earliest possible stage. The only distinct part of Chapter 3, compared to [LS_1], is the section on the opposite cocycles (Section 3.8). Some of the main results of Chapter 4 have been announced in [LS_2]; full versions including proofs will be published in papers [LS_3] (contents of Section 4.3), [LS_4] (Sections 4.1, 4.2, 4.4, 4.6, 4.8, Appendix A and parts of Section 4.5) and [Ska] (parts of Section 4.5, Section 4.7). The main difference between the treatment in the thesis and the papers mentioned above lies in that the latter are written in a more abstract language, with abstract operator spaces replacing concrete ones as the fundamental objects. Both approaches may be shown to be equivalent; the one adopted here permits certain simplifications of the proofs and remains closer to the traditional QS framework. Another important difference between the thesis and the papers mentioned above is that here we concentrate, whenever possible, on the class of completely bounded, everywhere defined processes, whereas in particular in [LS_3] the perspective is broadened to discuss processes which are densely defined, assuming only complete boundedness of their columns. Whilst analytically natural, this generality is not needed for the purposes of this thesis. The paper [LS_3] also contains a characterisation of those QS cocycles on finite dimensional operator spaces that arise as solutions of QS differential equations, similar in spirit to the characterisation for QS convolution cocycles in Section 3.4. Another difference is that proofs of various properties of solutions of QS differential equations in Section 4.3 are based on the uniqueness theorem, rather than on the explicit checks as in [LS_3].

Here the introduction ends, and the proper journey begins. Everyone is invited!

Chapter 2

Preliminaries

This chapter contains the preliminary facts and establishes some of the terminology needed in the sequel. After discussing the general notations used in the thesis, the standard notions of the theory of operator spaces, including the concept of the matrix spaces, are given. We proceed to describe the symmetric Fock space setup, main Fock space operators and define quantum stochastic processes. This is sufficient to understand the definition of QS cocycles and QS convolution cocycles as introduced in Section 4.2. The rest of the chapter is devoted to the quantum stochastic integration. The basic definitions are given and the Fundamental Formulas and the Fundamental Estimate included; finally the preliminary aspects of the iterated QS integrals are discussed.

2.1 General facts and notations

All vector spaces in this work are complex, scalar products are linear in the second variable. In general we use $V, W, A, \ldots$ for vector spaces (and other purely algebraic objects: algebras, coalgebras, etc.) and $\mathsf{V}, \mathsf{W}, \mathsf{A}, \ldots$ for operator spaces (C^*-algebras, operator space coalgebras, etc.); Hilbert spaces are usually denoted by letters h, k, H, K (k will be usually reserved for the noise dimension space, cf. Section 2.3). The vector space of linear maps between vector spaces V, W will be denoted $L(V; W)$, and the normed space of all bounded linear maps between normed spaces V, W will be denoted by $B(\mathsf{V}; \mathsf{W})$. If $S \subset V$, $\operatorname{Lin} S$ denotes the linear span of S.

For a subset V of an involutive vector space W, $V^\dagger := \{w \in W : w^\dagger \in V\}$. When both vector spaces V, W are equipped with involutions, and $\phi \in L(V; W)$, the map $\phi^\dagger \in L(V; W)$ is defined by $\phi^\dagger(v) = \phi(v^\dagger)^\dagger$, $v \in V$.

For a function $f : \mathbb{R}_+ \to \mathsf{h}$ and subinterval I of $\mathbb{R}_+$, f_I denotes the function $\mathbb{R}_+ \to \mathsf{h}$ which agrees with f on I and is zero outside I (cf. standard indicator-function notation). This convention also applies to vectors, by viewing them as constant functions—for example

$$\xi_{[s,t[}, \text{ for } \xi \in \mathsf{h} \text{ and } 0 \leq s < t.$$

Let h be a Hilbert space. Ampliations are denoted

$$\iota_{\mathsf{h}} : B(\mathsf{H}) \to B(\mathsf{H} \otimes \mathsf{h}),\ T \mapsto T \otimes I_{\mathsf{h}},$$

and each vector $\xi \in \mathsf{h}$ defines operators

$$E_\xi : \mathsf{H} \to \mathsf{H} \otimes \mathsf{h},\ v \mapsto v \otimes \xi \text{ and } E^\xi = (E_\xi)^*, \tag{2.1.1}$$

generalising Dirac's bra-ket notation:

$$E_\xi = I_{\mathsf{H}} \otimes |\xi\rangle \text{ and } E^\xi = I_{\mathsf{H}} \otimes \langle\xi|.$$

The particular Hilbert space H will always be clear from the context. When $\xi, \eta \in \mathsf{h}$, the functional $\omega_{\xi,\eta} : B(\mathsf{h}) \to \mathbb{C}$ is defined by

$$\omega_{\xi,\eta}(T) = \langle \xi, T\eta \rangle, \tag{2.1.2}$$

$T \in B(\mathsf{h})$. We will write simply ω_ξ for $\omega_{\xi,\xi}$ $(\xi \in \mathsf{h})$.

Now let D be a subset of k. The following notation will be employed:

$$\widehat{D} := \mathrm{Lin}\left\{\widehat{\xi} : \xi \in D\right\},\ \text{where } \widehat{\xi} := \begin{pmatrix} 1 \\ \xi \end{pmatrix} \in \widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}. \tag{2.1.3}$$

Whenever J is a set and $f : J \to \mathsf{k}$, the function $\hat{f} : J \to \widehat{\mathsf{k}}$ is defined by

$$\hat{f}(s) := \widehat{f(s)}, \quad s \in J.$$

The orthogonal projection from $\widehat{\mathsf{k}}$ onto k is denoted by Δ^{QS}; if $\mathsf{h}_1, \mathsf{h}_2$ are Hilbert spaces, $\Delta^{QS}_{\mathsf{h}_1,\mathsf{h}_2} := \mathrm{id}_{\mathsf{h}_1} \otimes \Delta^{QS} \otimes \mathrm{id}_{\mathsf{h}_2}$.

Let now E be a dense subspace of some Hilbert space h. $\mathcal{O}(E)$ will denote the vector space of linear operators in h with domain E, $\mathcal{O}^\dagger(E)$ the vector space of these operators in $\mathcal{O}(E)$ whose adjoints' domains include E. The

space $\mathcal{O}^\dagger(E)$ is equipped with the involution $\dagger$:

$$T^\dagger := T^*|_E, \quad T \in \mathcal{O}^\dagger(E). \tag{2.1.4}$$

In chapter 3 some further subspaces of $\mathcal{O}(E)$ will be specified. Usually when T is an operator defined on the whole of h its restriction to E will be denoted by the same letter; this notational abuse leads to the formal inclusion $B(\mathsf{h}) \subset \mathcal{O}(E)$. Note that $\mathcal{O}^\dagger(\mathsf{h}) = B(\mathsf{h})$ and the formula (2.1.2) (for $\xi \in \mathsf{h}$, $\eta \in E$) defines a linear functional on $\mathcal{O}(E)$, denoted also by $\omega_{\xi,\eta}$.

If $\mathcal{A}$ is a *-algebra, an element $a \in \mathcal{A}$ is said to be *positive* if $a = \sum_{i=1}^n a_i^* a_i$ for some $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathcal{A}$. The set $\mathcal{A}_+$ of positive elements is always a cone, but in general it may fail to generate the whole of $\mathcal{A}$ if $\mathcal{A}$ is nonunital (think of the algebra of all polynomials without constant term). By a positive map between *-algebras is understood a linear map preserving cones of positive elements.

At some point we will need a particular notation for converting a subset of the set $\{1, \ldots, n\}$ (denoted below by ν) into subsets of $\{1, \ldots, n+1\}$ ($n \in \mathbb{N}$) developed in [LW$_4$]:

$$\overset{\circ\!\rightarrow}{\nu} := \{1 + k : k \in \nu\} \quad \text{and} \quad \overset{\bullet\!\rightarrow}{\nu} := \{1\} \cup \overset{\circ\!\rightarrow}{\nu}. \tag{2.1.5}$$

Algebraic tensor product of vector spaces is denoted by $\odot$; for maps mainly the symbol $\otimes$ is used, meaning either algebraic tensor product of linear maps, or its relevant continuous extension. When there is no danger of confusion the symbol of tensor product between vectors is dropped (e.g. $u \otimes \varepsilon(f)$ is written as $u\varepsilon(f)$, the notation to be introduced in Section 2.3). The set of nonnegative integers is denoted $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. When S_1, S_2 are arbitrary sets, $S_1 \subset\subset S_2$ means that S_1 is a *finite* subset of S_2. The symbol $\Delta_n[0,t]$ denotes the simplex $\{\boldsymbol{s} \in [0,t]^n \mid s_n \geq \cdots \geq s_1\}$; for many estimates in the main text it is worth recalling that the Lebesgue measure of $\Delta_n[0,t]$ is $\frac{t^n}{n!}$. The symbol τ is often used for tensor flips both between Hilbert spaces ($\mathsf{h}_1 \otimes \mathsf{h}_2 \to \mathsf{h}_2 \otimes \mathsf{h}_1$) and algebras of bounded operators ($B(\mathsf{h}_1 \otimes \mathsf{h}_2) \to B(\mathsf{h}_2 \otimes \mathsf{h}_1)$) - note that the latter

is a *-isomorphism. When $\mathsf{A} \subset B(\mathsf{h})$ is a nondegenerate C^*-algebra, A'' (the bicommutant of A) equals the strong closure of A - this is the content of von Neumann's Double Commutant Theorem.

By a *nonnegative-definite kernel* on a non-empty set S is understood a map $k : S \times S \to \mathbb{C}$ such that for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $s_1, \ldots, s_n \in S$ there is $\sum_{i,j=1}^n \overline{\lambda_i} \lambda_j k(s_i, s_j) \geq 0$. A pair (K, η), where K is a Hilbert space and $\eta : S \to \mathsf{K}$ is called a *(minimal) Kolmogorov construction* for the kernel k if

$$k(s,t) = \langle \eta(s), \eta(t) \rangle, \quad s, t \in S \quad \text{and} \quad \overline{\mathrm{Lin}}\, \eta(S) = \mathsf{K}.$$

A minimal Kolmogorov construction exists for each nonnegative-definite kernel and is unique up to a unitary transformation. The notion of nonnegative definite maps has a natural generalisation to maps on $S \times S$ taking values in a C^*-algebra; there is also an obvious counterpart of the notion of Kolmogorov construction for such maps.

Quantum stochastic is usually abbreviated to QS, operator space to OS, completely positive to CP and completely positive and contractive to CPC.

2.2 Operator spaces

For general introduction to the theory of operator spaces we refer to [EfR]; for a functional-analytic point of view on this theory, with numerous applications to operator algebras, we recommend [Pis$_2$]. Let us recall that an (abstract) operator space V is a Banach space equipped with norms on matrix spaces over V ($M_n(\mathsf{V})$, $n \in \mathbb{N}$), satisfying so-called Ruan's axioms. If V, W are operator spaces and $\phi : \mathsf{V} \to \mathsf{W}$ is a linear map, ϕ is called *completely bounded* if

$$\|\phi\|_{cb} := \sup_{n \in \mathbb{N}} \|\phi^{(n)}\| < \infty,$$

where $\phi^{(n)} : M_n(\mathsf{V}) \to M_n(\mathsf{W})$ is an obvious matricial extension of ϕ - ϕ is applied to each entry of the matrix. Analogously we define completely contractive and completely isometric maps. The space of completely bounded

maps between operator spaces V and W is itself an operator space, denoted further by $CB(\mathsf{V};\mathsf{W})$ - the matrix norms are introduced with the help of the algebraic identification $M_n(CB(\mathsf{V};\mathsf{W})) \cong CB(\mathsf{V};M_n(\mathsf{W}))$. In particular, as every bounded linear functional is completely bounded, we obtain an operator space structure on $\mathsf{V}^* = CB(\mathsf{V};\mathbb{C})$.

For this work it will usually be sufficient to work with concrete operator spaces, i.e. closed subspaces of the space $B(\mathsf{h}_1;\mathsf{h}_2)$ for Hilbert spaces $\mathsf{h}_1, \mathsf{h}_2$, with matricial norms induced by the identification $M_n(B(\mathsf{h}_1;\mathsf{h}_2)) \cong B(\mathsf{h}_1^{\oplus n};\mathsf{h}_2^{\oplus n})$. Ruan's Theorem says that each abstract operator space is completely isometric to a concrete operator space.

The spatial (or minimal) tensor product of operator spaces $\mathsf{V} \subset B(\mathsf{h}_1)$ and $\mathsf{W} \subset B(\mathsf{h}_2)$ is the norm closure of $\mathsf{V} \odot \mathsf{W}$ in $B(\mathsf{h}_1 \otimes \mathsf{h}_2)$. It is denoted by $\mathsf{V} \otimes \mathsf{W}$, and in fact does not depend on concrete representations of V and W - it is completely isometric to the norm closure of the canonical image of $\mathsf{V} \odot \mathsf{W}$ in $CB(\mathsf{V}^*;\mathsf{W})$. The important fact for us is that when V_1, V_2, W_1, W_2 are operator spaces, $\phi_1 \in CB(\mathsf{V}_1;\mathsf{W}_1)$ and $\phi_2 \in CB(\mathsf{V}_2;\mathsf{W}_2)$, then the map $\phi_1 \odot \phi_2 : \mathsf{V}_1 \odot \mathsf{V}_2 \to \mathsf{W}_1 \odot \mathsf{W}_2$ has a (unique) completely bounded extension $\phi_1 \otimes \phi_2 : \mathsf{V}_1 \otimes \mathsf{V}_2 \to \mathsf{W}_1 \otimes \mathsf{W}_2$, satisfying $\|\phi_1 \otimes \phi_2\|_{cb} = \|\phi_1\|_{cb}\|\phi_2\|_{cb}$.

Let A, B be C^*-algebras (so in particular operator spaces) and $\phi : \mathsf{A} \to \mathsf{B}$ be linear. The map ϕ is called completely positive (CP) if for all $n \in \mathbb{N}$ $\phi^{(n)} : M_n(\mathsf{A}) \to M_n(\mathsf{B})$ is positive. Every completely positive map $\phi : \mathsf{A} \to \mathsf{B}$ is completely bounded and its complete bound coincides with its operator bound; if A is unital, $\|\phi\| = \|\phi(I)\|$.

Matrix spaces

We need a concept of matrix spaces introduced by J.M. Lindsay and S.J. Wills in [LW$_3$]. Let $\mathsf{V} \subset B(\mathsf{K})$ be an operator space and let h be a supplementary Hilbert space. The (concrete) operator space:

$$M_{\mathsf{h}}(\mathsf{V}) = \{T \in B(\mathsf{K} \otimes \mathsf{h}) : E^c T E_d \in \mathsf{V} \text{ for all } c, d \in \mathsf{h}\}$$

is called an h-*matrix space* over V. It is easy to see that $M_{\mathsf{h}}(\mathsf{V})$ contains the spatial tensor product $\mathsf{V} \otimes B(\mathsf{h})$. If V is ultraweakly closed, $M_{\mathsf{h}}(\mathsf{V})$ coincides with the ultraweak closure of $\mathsf{V} \odot B(\mathsf{h})$, denoted by $\mathsf{V}\overline{\otimes}B(\mathsf{h})$. If h has finite dimension, n say, then $M_{\mathsf{h}}(\mathsf{V}) \cong M_n(\mathsf{V})$. In general both inclusions $\mathsf{V}\otimes B(\mathsf{h}) \subset M_{\mathsf{h}}(\mathsf{V}) \subset \mathsf{V}\overline{\otimes}B(\mathsf{h})$ are proper.

Later the following functorial property of matrix spaces will be needed. If h_1, h_2 are Hilbert spaces then there is a canonical complete isometry

$$M_{\mathsf{h}_1}\left(M_{\mathsf{h}_2}(\mathsf{V})\right) \cong M_{\mathsf{h}_2\otimes\mathsf{h}_1}(\mathsf{V}). \tag{2.2.1}$$

The latter space is also completely isometric to $M_{\mathsf{h}_1\otimes\mathsf{h}_2}(\mathsf{V})$, with the complete isometry implemented by the tensor flip $\mathsf{K}\otimes\mathsf{h}_2\otimes\mathsf{h}_1$ to $\mathsf{K}\otimes\mathsf{h}_1\otimes\mathsf{h}_2$.

Whenever $\mathsf{W} \subset B(\mathsf{H})$ is another operator space, and $\phi \in CB(\mathsf{V};\mathsf{W})$, the map $\phi\otimes \mathrm{id}_{B(\mathsf{h})}$ extends uniquely to a completely bounded map $\phi^{(\mathsf{h})} : M_{\mathsf{h}}(\mathsf{V}) \to M_{\mathsf{h}}(\mathsf{W})$ satisfying

$$E^c\big(\phi^{(\mathsf{h})}(T)\big)E_d = \phi(E^cTE_d),$$

for all $T \in M_{\mathsf{h}}(\mathsf{V})$, $c, d \in \mathsf{h}$. The map $\phi^{(\mathsf{h})}$ is called the h-*lifting* of ϕ and satisfies $\|\phi^{(\mathsf{h})}\|_{cb} = \|\phi\|_{cb}$. Observe that when $\mathsf{h} = \mathbb{C}^n$ ($n \in \mathbb{N}$), the map $\phi^{(\mathsf{h})}$ is just the previously introduced lifting $\phi^{(n)} : M_n(\mathsf{V}) \to M_n(\mathsf{W})$.

Analogously one can define h-*column space* and h-*row space* over V, denoted respectively by $C_{\mathsf{h}}(\mathsf{V})$ and $R_{\mathsf{h}}(\mathsf{V})$ (so that for example $C_{\mathsf{h}}(\mathsf{V})$ is a subspace of $B(\mathsf{h};\mathsf{K}\otimes\mathsf{h})$). For $\mathsf{V} = \mathbb{C}$ the notation $|\mathsf{h}\rangle := C_{\mathsf{h}}(\mathbb{C})$, $\langle\mathsf{h}| := R_{\mathsf{h}}(\mathbb{C})$ is also used. Completely bounded maps lift to those as well, and the liftings are denoted respectively by superscripts $^{(|\mathsf{h}\rangle)}$, $^{(\langle\mathsf{h}|)}$.

2.3 Fock space notations and QS processes

In this section the definition of a symmetric Fock space is recalled and the terminology concerning quantum stochastic processes established.

Symmetric Fock space

Recall that the symmetric Fock space over a Hilbert space h is defined as the orthogonal Hilbert space sum $\bigoplus_{n=0}^{\infty}(\mathsf{h}^{\otimes n}_{\mathrm{sym}})$, where $\mathsf{h}^{\otimes 0}_{\mathrm{sym}} := \mathbb{C}$, and for each $n \in \mathbb{N}$ the space $\mathsf{h}^{\otimes n}_{\mathrm{sym}}$ is a closed subspace of $\mathsf{h}^{\otimes n}$ generated by vectors $\{u^{\otimes n} : u \in \mathsf{h}\}$. It will be denoted by $\Gamma(\mathsf{h})$. By the exponential property of Fock space is understood the existence of the canonical isomorphism $\Gamma(\mathsf{h}_1 \oplus \mathsf{h}_2) \cong \Gamma(\mathsf{h}_1) \otimes \Gamma(\mathsf{h}_2)$.

Let k be a Hilbert space, called the *noise dimension space*. There is a natural isomorphism $L^2(\mathbb{R}_+) \otimes \mathsf{k} \cong L^2(\mathbb{R}_+; \mathsf{k})$, where the second space is a space of square integrable functions on $\mathbb{R}^+$ with values in k. Integrability of Banach-space-valued functions is always understood as Bochner integrability. The stochastic arena for the action of QS stochastic cocycles, $\Gamma(L^2(\mathbb{R}_+; \mathsf{k}))$, will be denoted by $\mathcal{F}_{\mathsf{k}}$ (or $\mathcal{F}$ if the space k is clear from the context). Whenever J is a subinterval of $\mathbb{R}_+$ we will write $\mathcal{F}_{J;\mathsf{k}}$ (or $\mathcal{F}_J$) for $\Gamma(L^2(J; \mathsf{k}))$. The exponential property of Fock space gives $\mathcal{F}$ a structure of a product system in the sense of Arveson ([Arv], [Bha]):

$$\mathcal{F} = \mathcal{F}_{[0,s)} \otimes \mathcal{F}_{[s,s+t)} \otimes \mathcal{F}_{[s+t,\infty)}, \quad s, t \geq 0. \tag{2.3.1}$$

The exponential vectors in $\mathcal{F}$ are defined by

$$\varepsilon(f) := \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} f^{\otimes n}, \quad f \in L^2(\mathbb{R}_+; \mathsf{k}).$$

The terminology is justified by the following relation ($f, g \in L^2(\mathbb{R}_+; \mathsf{k})$):

$$\langle \varepsilon(f), \varepsilon(g) \rangle = \exp(\langle f, g \rangle),$$

moreover the exponential vectors behave well under the tensor decomposition of $\mathcal{F}$, given in (2.3.1):

$$\varepsilon(f) = \varepsilon(f_{[0,s)}) \otimes \varepsilon(f_{[s,s+t)}) \otimes \varepsilon(f_{[s+t,\infty)}),$$

$s, t \geq 0$, $f \in L^2(\mathbb{R}_+; \mathsf{k})$. The vector $\varepsilon(0)$ is denoted by Ω and called the *vacuum vector*; the vector state ω_Ω is called the *vacuum state* on $B(\mathcal{F})$. Observe that $\mathcal{F}_{[s,s+t)}$ is viewed as a subspace of $\mathcal{F}$ via the identification

$$\varepsilon(f) \longrightarrow \varepsilon(0_{[0,s)}) \otimes \varepsilon(f) \otimes \varepsilon(0_{[s+t,\infty)}),$$

$f \in L^2([s, s+t); \mathsf{k})$.

For any dense subspace $D \subset \mathsf{k}$ define

$$\mathbb{S}_D := \mathrm{Lin}\{d_{[0,s)} : d \in D, s \in \mathbb{R}_+\}$$

($\mathbb{S} := \mathbb{S}_\mathsf{k}$) and corresponding subspaces of $\mathcal{F}$:

$$\mathcal{E}_D := \mathrm{Lin}\{\varepsilon(f) : f \in \mathbb{S}_D\}, \quad \mathcal{E} := \mathcal{E}_\mathsf{k}.$$

It may be shown that $\mathcal{E}_D$ is a dense subspace of $\mathcal{F}$ (see for example [Lin]). Elements of $\mathcal{E}_D$ will play the role of test functions for QS processes (notice that our step functions are right-continuous).

The basic operators for the QS integration are the *conservation* (known also as *preservation* or *gauge*), *creation* and *annihilation* operators, defined on $\mathcal{E}$ respectively by

$$\begin{aligned} n(T)\varepsilon(f) &= \frac{d}{dt}\varepsilon(\mathrm{e}^{tT} f)|_{t=0} \\ a^\dagger(g)\varepsilon(f) &= \frac{d}{dt}\varepsilon(f + tg)|_{t=0} \\ a(g)\varepsilon(f) &= \langle g, f \rangle \varepsilon(f), \end{aligned}$$

where $f, g \in \mathbb{S}$ and $T \in B(L^2(\mathbb{R}_+; \mathsf{k}))$.

Denote (for $t \geq 0$) by s_t the shift isometry acting from $L^2(\mathbb{R}_+; \mathsf{k})$ to $L^2([t, \infty); \mathsf{k})$ defined by

$$(s_t f)(s) = f(s - t), \quad s \geq t.$$

This can be 'second-quantised' to the isometry $S_t : \mathcal{F} \to \mathcal{F}_{[t,\infty)}$, which is the continuous extension of the map defined on $\mathcal{E}$ by

$$S_t(\varepsilon(f)) = \varepsilon(s_t f).$$

The *CCR flow of index* k is the E_0-semigroup ([Arv]) $\{\sigma_t : t \geq 0\}$ on $B(\mathcal{F})$ defined by

$$\sigma_t(X) = I_{\mathcal{F}_{[0,t)}} \otimes (S_t X S_t^*)\,, \tag{2.3.2}$$

$X \in B(\mathcal{F})$. Observe that the above formula may be extended to all operators $X \in \mathcal{O}(\mathcal{E}_D)$, in which case $\sigma_t(X)$ also belongs to $\mathcal{O}(\mathcal{E}_D)$.

When k_0 is a closed subspace of a Hilbert space k, by the k_0-*vacuum conditional expectation* is understood a projection $\mathbb{E}_{\mathcal{F}_{\mathsf{k}_0}} : B(\mathcal{F}_{\mathsf{k}}) \to B(\mathcal{F}_{\mathsf{k}_0})$ preserving the vacuum state. Its action is given by

$$T \mapsto E^{\Omega_0} T E_{\Omega_0},$$

where Ω_0 is the vacuum vector in $\mathcal{F}_{\mathsf{k}_0^\perp}$.

It is often useful to view the symmetric Fock space from a different point, using the description due to A. Guichardet ([Gui]). For a subinterval $J \subset \mathbb{R}_+$ and $n \in \mathbb{N}_0$ put

$$\Gamma_J = \{\sigma \subset J : \#\sigma < \infty\}, \text{ and } \Gamma_J^{(n)} = \{\sigma \in \Gamma_J : \#\sigma = n\}.$$

Observe that (for $n \in \mathbb{N}$) $\Gamma_J^{(n)}$ can be naturally identified with the set $\{(t_1, \ldots, t_n) \in J^n : t_1 < \ldots < t_n\}$, and so is equipped with a measure induced by n-dimensional Lebesgue measure. Using the fact that Γ_J is a disjoint union $\bigcup_{n=0}^\infty \Gamma_J^{(n)}$ and declaring $\Gamma_J^{(0)} = \{\emptyset\}$ to be an atom of the unit measure allows us to introduce the *Guichardet measure* (or the *symmetric measure*) on

Γ_J as the sum of the measures described above. Define

$$\mathcal{G}_{J,\mathsf{k}} := \left\{ F \in L^2\left(\Gamma_J; \bigoplus_{n=0}^{\infty} \mathsf{k}^{\otimes n}\right) : F(\sigma) \in \mathsf{k}^{\otimes \# \sigma} \text{ for a.e. } \sigma \right\},$$

where $\mathsf{k}^{\otimes 0} := \mathbb{C}$. For any $f \in L^2(J; \mathsf{k})$ a *product function* $\pi_f \in \mathcal{G}_{J,\mathsf{k}}$ is defined by

$$\pi_f(\sigma) = \begin{cases} f(s_n) \otimes \cdots \otimes f(s_1) & \text{if } \sigma = \{s_1 < \ldots < s_n\} \\ 1 & \text{if } \sigma = \{\emptyset\}. \end{cases}$$

The crucial fact here is that each product function belongs to $\mathcal{G}_{J,\mathsf{k}}$ and the map $\pi_f \to \varepsilon(f)$ extends uniquely to a Hilbert space isomorphism $\mathcal{G}_{J,\mathsf{k}} \to \mathcal{F}_{J,\mathsf{k}}$.

Processes

Let h be an additional Hilbert space with a dense subspace E and fix a dense subspace $D \subset \mathsf{k}$. By an h-*operator process* (with the domain $\mathcal{D} := E \odot \mathcal{E}_D$) we understand a family $X = \big(X_t\big)_{t \geq 0}$ of operators on $\mathsf{h} \otimes \mathcal{F}$, each having the (dense) domain $\mathcal{D}$, being weak-operator measurable in t and adapted to the natural Fock-space operator-filtration. Thus $X : \mathbb{R}_+ \to \mathcal{O}(E \odot \mathcal{E}_D)$, $t \mapsto X_t\xi$ is weakly measurable for all $\xi \in \mathcal{D}$ and, for each $t \geq 0$, $\zeta \in E$, $f \in \mathbb{S}_D$,

$$X_t(\zeta \otimes \varepsilon(f)) = X(t)\big(\zeta \otimes \varepsilon(f_{[0,t[})\big) \otimes \varepsilon(f_{[t,\infty[})$$

for some operator $X(t) \in \mathcal{O}\big(E \odot \mathcal{E}_{D,[0,t)}\big)$, where $\mathbb{S}_{D,[0,t)}$ is defined as $\mathbb{S}$ is, except that $\mathbb{R}_+$ is replaced by $[0,t)$. Two h-operator processes X, X' with the domain $\mathcal{D}$ are called *indistinguishable* if for all $\xi \in \mathcal{D}$, $\zeta \in \mathsf{h} \otimes \mathcal{F}$,

$$\langle \zeta, X_t\xi \rangle = \langle \zeta, X'_t\xi \rangle \quad \text{for almost all } t \in \mathbb{R}_+.$$

Identifying indistinguishable processes allows us to consider the linear space of all h-operator processes with the domain $E \odot \mathcal{E}_D$, denoted $\mathbb{P}(E, \mathcal{E}_D)$ (or $\mathbb{P}(\mathcal{E}_D)$ if $\mathsf{h} = \mathbb{C}$).

A process $X \in \mathbb{P}(E, \mathcal{E}_D)$ is called *weakly regular* if for each $f, g \in \mathbb{S}_D$, $t \geq 0$

the operator $E^{\varepsilon(f)}X_tE_{\varepsilon(g)} : E \to \mathsf{h}$ is bounded (recall the notation (2.1.1)), and, for fixed f, g, the respective norms are locally bounded with respect to t. It is called *weakly continuous* if for each $\zeta, \xi \in \mathcal{D}$ the scalar-valued function $t \to \langle \zeta, X_t\xi \rangle$ is continuous. Further X as above is called *square integrable* if for each $\xi \in \mathcal{D}$ the vector-valued function $t \to X_t\xi$ is locally square integrable; it is called *locally bounded* if each $t \to X_t\xi$ is locally bounded, *continuous* if each $t \to X_t\xi$ is continuous and finally *Hölder continuous* if $t \to X_t\xi$ is Hölder continuous at 0 with exponent $\frac{1}{2}$:

$$\limsup_{t\to 0^+} t^{-\frac{1}{2}} \|X_t\xi - X_0\xi\| < \infty. \tag{2.3.3}$$

It is called *bounded* if X_t is a bounded operator for each $t \geq 0$. In such case we will denote the continuous extension of X_t to an operator in $B(\mathsf{h} \otimes \mathcal{F})$ by the same letter.

The respective subspaces of $\mathbb{P}(E, \mathcal{E}_D)$ will be denoted by $\mathbb{P}_{\mathrm{wr}}(E, \mathcal{E}_D)$ (weakly regular), $\mathbb{P}_{\mathrm{wc}}(E, \mathcal{E}_D)$ (weakly continuous), $\mathbb{P}_2(E, \mathcal{E}_D)$ (square integrable), $\mathbb{P}_{\mathrm{lb}}(E, \mathcal{E}_D)$ (locally bounded), $\mathbb{P}_{\mathrm{c}}(E, \mathcal{E}_D)$ (continuous), $\mathbb{P}_{\mathrm{Hc}}(E, \mathcal{E}_D)$ (Hölder continuous) and $\mathbb{P}_{\mathrm{b}}(E, \mathcal{E}_D)$ (bounded). Note that

$$\mathbb{P}_{\mathrm{wc}}(\mathcal{E}_D) \subset \mathbb{P}_{\mathrm{wr}}(\mathcal{E}_D). \tag{2.3.4}$$

Additionally let

$$\mathbb{P}^\dagger(\mathsf{h}, \mathcal{E}_D) := \{X \in \mathbb{P}(\mathsf{h}, \mathcal{E}_D) : \forall_{t \in 0}\ X_t \in \mathcal{O}^\dagger(\mathsf{h} \odot \mathcal{E}_D)\},$$

$$\mathbb{P}^\dagger_\alpha(\mathsf{h}, \mathcal{E}_D) := \{X \in \mathbb{P}^\dagger(\mathsf{h}, \mathcal{E}_D) : X, X^\dagger \in \mathbb{P}_\alpha(\mathsf{h}, \mathcal{E}_D)\},$$

where α may be any of the above subscripts, and the involution $X \to X^\dagger$ is induced pointwise from the involution (2.1.4).

Let now V be a vector space and $\mathsf{W} \subset B(\mathsf{h})$ an operator space. A linear map k from V to $\mathbb{P}(\mathsf{h}, \mathcal{E}_D)$ is called a *process on V with values in* W (and domain $\mathsf{h} \odot \mathcal{E}_D$) if for each $f, g \in \mathbb{S}_D$, $t \geq 0$, $v \in V$ operator $E^{\varepsilon(f)}k_t(v)E_{\varepsilon(g)}$ belongs to W. The vector space of all processes on V with values in W is written $\mathbb{P}(V; \mathsf{W}, \mathcal{E}_D)$

(or simply $\mathbb{P}(V;\mathcal{E}_D)$ if $\mathsf{W}=\mathbb{C}$). We say that $k\in\mathbb{P}(V;\mathsf{W},\mathcal{E}_D)$ is *weakly regular* (respectively weakly continuous, square integrable, etc.) if each $k(v)$ $(v\in V)$ satisfies the required property; similarly $\mathbb{P}^\dagger_\alpha(V;\mathsf{W},\mathcal{E}_D)$ denotes the space of all $k\in\mathbb{P}(V;\mathsf{W},\mathcal{E}_D)$ such that each $k(v)$ $(v\in V)$ belongs to $\mathbb{P}^\dagger_\alpha(\mathsf{h},\mathcal{E}_D)$ and for each $f,g\in\mathbb{S}_D$, $t\geq 0$, $v\in V$ operator $E^{\varepsilon(f)}k_t(v)^\dagger E_{\varepsilon(g)}$ belongs to W (again α is any of the available subscripts). If V is a subspace of some involutive vector space, each $k\in\mathbb{P}^\dagger_\alpha(V;\mathsf{W},\mathcal{E}_D)$ determines a process $k^\dagger\in\mathbb{P}^\dagger_\alpha(V^\dagger;\mathsf{W}^*,\mathcal{E}_D)$ by the standard formula $k^\dagger(v^\dagger):=(k(v))^\dagger$. If V is closed under the involution $\dagger$, W is selfadjoint and $k=k^\dagger$, k is called *real.*

If V is an operator space, $k\in\mathbb{P}(\mathsf{V};\mathsf{W},\mathcal{E}_D)$ is called *completely bounded* if for each $t\geq 0$, $v\in\mathsf{V}$ the process $k(v)$ is bounded and the resulting map $k_t:\mathsf{V}\to B(\mathsf{h}\otimes\mathcal{F})$ is completely bounded. It is called *weakly bounded* if for each $t\geq 0$ $f,g\in\mathbb{S}_D$ the map

$$\mathsf{V}\ni v\mapsto E^{\varepsilon(f)}k_t(v)E_{\varepsilon(g)}\in\mathsf{W}$$

is bounded, and *uniformly weakly bounded* if it is weakly bounded and for all $f,g\in\mathbb{S}_D$, $T>0$

$$\sup_{t<T}\|E^{\varepsilon(f)}k_t(\cdot)E_{\varepsilon(g)}\|<\infty.$$

Observe that due to Banach-Steinhaus Theorem processes which are weakly regular and weakly bounded are automatically uniformly weakly bounded.

Let $k\in\mathbb{P}(\mathsf{V};\mathsf{W},\mathcal{E})$. If for all $t\geq 0$, $g\in\mathbb{S}$ and $v\in\mathsf{V}$ the operator $k_t(v)E_{\varepsilon(g)}$ is in $C_{\mathcal{F}}(\mathsf{W})$ (equivalently, is bounded) and the map

$$\mathsf{V}\ni v\mapsto k_t(v)E_{\varepsilon(g)}\in C_{\mathcal{F}}(\mathsf{W}) \tag{2.3.5}$$

(denoted further by $k_{t,\varepsilon(g)}$) is bounded then k is said to have *bounded columns.* If k has bounded columns and for all $g\in\mathbb{S}$, $T\geq 0$

$$\sup_{t<T}\|k_{t,\varepsilon(g)}\|<\infty, \tag{2.3.6}$$

then k is called *strongly regular*. If the map $k_{t,\varepsilon(g)}$ is completely bounded (i.e. k

has *completely bounded columns*) and the estimate (2.3.6) holds for cb-norms, k is called cb-*strongly regular*. Similarly by analogy with (2.3.3) we can define cb-*Hölder continuous processes*.

The families of all processes satisfying conditions introduced above are denoted respectively by $\mathbb{P}_{\mathrm{cb}}(\mathsf{V};\mathsf{W},\mathcal{E})$ (completely bounded), $\mathbb{P}_{\mathrm{wb}}(\mathsf{V};\mathsf{W},\mathcal{E}_D)$ (weakly bounded), $\mathbb{P}_{\mathrm{uwb}}(\mathsf{V};\mathsf{W},\mathcal{E}_D)$ (uniformly weakly bounded), $\mathbb{P}_{\mathrm{sr}}(\mathsf{V};\mathsf{W},\mathcal{E}_D)$ (strongly regular) and $\mathbb{P}_{\mathrm{cbsr}}(\mathsf{V};\mathsf{W},\mathcal{E}_D)$ (cb-strongly regular). *Completely positive, unital and contractive* processes on A with values in B (where A, B are C^*-algebras) are defined by analogy with completely bounded processes.

2.4 Quantum stochastic integrals

Fix again a dense subspace D of the noise dimension space k, and let E be a dense subspace of another Hilbert space h. Recall the notation (2.1.3).

Quantum stochastic integration introduced by R.L. Hudson and K.R. Parthasarathy gives a linear map

$$\mathbb{P}_2(E \odot \widehat{D}, \mathcal{E}_D) \to \mathbb{P}_{\mathrm{c}}(E, \mathcal{E}_D),$$

denoted

$$X \to \int_0^{\cdot} X_s \, d\Lambda_s.$$

In the matrix form, if

$$X = \begin{bmatrix} K & G \\ F & H \end{bmatrix},$$

for certain processes K, G, F and H, the QS integral is given by:

$$\int_0^{\cdot} X_s \, d\Lambda_s = T(K) + A^*(F) + N(H) + A(G),$$

where T denotes the *time integral*, A^* the *creation integral*, A the *annihilation integral* and N the *preservation integral*. The latter integrals can be defined either by suitable limits of integrals of piecewise constant processes with respect

to integrators given in terms of the basic operators on Fock space introduced in the previous section ([Par], [Hud]), or by using the gradient operator of Malliavin calculus and Hitsuda-Skorohod integral ([Lin]). In the context of this work, where we are concerned with adapted, square-integrable processes defined on exponential domains, both definitions are equivalent. An advantage of the second (Malliavin-style) technique lies in its explicit independence on the choice of an orthonormal basis in k. The square integrability assumptions on processes in the entries of the matrix above guranteeing the existence of the QS integral may be weakened ([Lin]).

Quantum stochastic integral enjoys the *Fundamental Formulae* and the *Fundamental Estimate* below.

Theorem 2.4.1 (First Fundamental Formula). *Let* $X \in \mathbb{P}_2(E \odot \widehat{D}, \mathcal{E}_D)$, $\xi, \eta \in E$, $f, g \in \mathbb{S}_D$ *and* $t \geq 0$. *Then*

$$\left\langle \xi\varepsilon(f), \left(\int_0^t X_s d\Lambda_s\right) \eta\varepsilon(g) \right\rangle = \int_0^t \left\langle \xi\hat{f}(s)\varepsilon(f), X_s\left(\eta\hat{g}(s)\varepsilon(g)\right) \right\rangle ds. \tag{2.4.1}$$

As $E \odot \mathcal{E}_D$ is dense in $\mathsf{h} \otimes \mathcal{F}$, the First Fundamental Formula determines $\int_0^{\cdot} X_s \, d\Lambda_s$ uniquely. It also implies that for $X \in \mathbb{P}_2^\dagger(E \odot \widehat{D}, \mathcal{E}_D)$, $t \geq 0$,

$$\left(\int_0^t X_s d\Lambda_s\right)^\dagger = \int_0^t X_s^\dagger d\Lambda_s. \tag{2.4.2}$$

The next estimate is very important for constructing solutions of QS differential equations. We use a natural notation $\int_r^t X_s d\Lambda_s = \int_0^t X_s d\Lambda_s - \int_0^r X_s d\Lambda_s$ $(0 \leq r \leq t)$.

Theorem 2.4.2 (Fundamental Estimate). *Let* $X \in \mathbb{P}_2(E \odot \widehat{D}, \mathcal{E}_D)$, $\xi \in E$, $f \in \mathbb{S}_D$ *and* $0 \leq r \leq t \leq T$. *Then*

$$\left\| \left(\int_r^t X_s d\Lambda_s\right) \xi\varepsilon(g) \right\|^2 \leq C(f,T) \int_r^t \|X_s\left(\xi\hat{f}(s)\varepsilon(f)\right)\|^2 ds, \tag{2.4.3}$$

where $C(f,T) > 0$ *is a constant dependent only on* f *and* T.

Finally the Second Fundamental Formula is a QS extension of the Itô Formula.

Theorem 2.4.3 (Second Fundamental Formula). *Let* $X, Y \in \mathbb{P}_2(E \odot \widehat{D}, \mathcal{E}_D)$, $\xi, \eta \in E$, $f, g \in \mathbb{S}_D$, $t \geq 0$ *and let* $L = \int_0^{\cdot} X_s \, d\Lambda_s$, $M = \int_0^{\cdot} Y_s \, d\Lambda_s$. *Then*

(2.4.4)

$$\begin{aligned}\langle L_t(\xi\varepsilon(f)), M_t(\eta\varepsilon(g))\rangle = \int_0^t ds \Big(& \Big\langle \widetilde{L_s}\Big(\xi \hat{f}(s)\varepsilon(f)\Big), Y_s(\eta\hat{g}(s)\varepsilon(g))\Big\rangle \\ & + \Big\langle X_s\Big(\xi\hat{f}(s)\varepsilon(f)\Big), \widetilde{M}_s(\eta\hat{g}(s)\varepsilon(g))\Big\rangle \\ & + \Big\langle X_s\Big(\xi\hat{f}(s)\varepsilon(f)\Big), \Delta^{QS}_{\mathsf{h},\mathcal{F}} Y_s(\eta\hat{g}(s)\varepsilon(g))\Big\rangle\Big),\end{aligned}$$

where $\widetilde{L}_t$, $\widetilde{M}_t \in \mathcal{O}(E \odot \widehat{D} \odot \mathcal{E}_D)$ *are given respectively by* $\widetilde{L}_t := \tau\big(I_{\widehat{D}} \otimes L_s\big)$, $\widetilde{M}_t := \tau\big(I_{\widehat{D}} \otimes M_s\big)$, *and* τ *is the tensor flip:* $\tau : \mathcal{O}(\widehat{D} \odot E \odot \mathcal{E}_D) \to \mathcal{O}(E \odot \widehat{D} \odot \mathcal{E}_D)$.

Iterated QS integration

For $L \in \mathcal{O}(E \odot \widehat{D}^{\odot n})$, where E is a dense subspace of a Hilbert space h, define $\Lambda^n(L) \in \mathbb{P}_c(E, \mathcal{E}_D)$ recursively as follows (we write $\Lambda^n_t(L)$ for $\Lambda^n(L)_t$):

$$\Lambda^0_t(L) = L \otimes I_{\mathcal{E}_D}, \text{ and, for } n \geq 1, \ \Lambda^n_t(L) = \int_0^t \Lambda^{n-1}_s(L) \, d\Lambda_s,$$

by viewing $E \odot \widehat{D}^{\odot n}$ as $(E \odot \widehat{D}) \odot \widehat{D}^{\odot(n-1)}$.

Letting ξ, η, f, g and T be as in the Fundamental Formulae above, we obtain

$$\langle \xi\varepsilon(f), \Lambda^n_t(L)\eta\varepsilon(g)\rangle = \int_{\Delta_n[0,t]} \langle \zeta(\boldsymbol{s}), (L \otimes I_{\mathcal{F}})\theta(\boldsymbol{s})\rangle \, d\boldsymbol{s}, \tag{2.4.5}$$

and

$$\|\Lambda^n_t(L)\xi\varepsilon(f)\|^2 \leq C(f,T)^n \int_{\Delta_n[0,t]} \|(L \otimes I_{\mathcal{F}})\zeta(\boldsymbol{s})\|^2 \, d\boldsymbol{s}, \tag{2.4.6}$$

where $\zeta(\boldsymbol{s}) := \xi\hat{f}^{\otimes n}(\boldsymbol{s})\varepsilon(f)$ ($\hat{f}^{\otimes n}(\boldsymbol{s}) := \hat{f}(s_n)\cdots\hat{f}(s_1)$) and similarly $\theta(\boldsymbol{s}) := \eta\hat{g}^{\otimes n}(\boldsymbol{s})\varepsilon(g)$.

Further we would like to describe the action of the sequences of iterated QS integrals. To this end we introduce spaces of so-called *matrix-sum kernels*. Let E be again a dense subspace of a Hilbert space h and let

$$\mathscr{S}_{E,D} := \{F = (F_n)_{n=0}^{\infty} \,:\, \forall_{n\in\mathbb{N}_0}\ F_n \in \mathcal{O}(E \odot \widehat{D}^{\odot n})\},$$

$$\mathscr{S}_{E,D}^{\dagger} := \left\{F \in \mathscr{S}_{E,D} \,:\, \forall_{n\in\mathbb{N}_0}\, F_n \in \mathcal{O}^{\dagger}(E \odot \widehat{D}^{\odot n})\right\},$$

(where as usual $\widehat{D}^{\odot 0}$ denotes $\mathbb{C}$). We also write $\mathscr{S}_D := \mathscr{S}_{\mathbb{C},D}$, $\mathscr{S}_D^{\dagger} = \mathscr{S}_{\mathbb{C},D}^{\dagger}$. It is obvious that $\mathscr{S}_{E,D}^{\dagger}$ is a subspace of a vector space $\mathscr{S}_{E,D}$. Moreover $\mathscr{S}_{E,D}^{\dagger}$ is an involutive vector space with the involution † induced pointwise from the involution (2.1.4).

For QS purposes we need to impose a growth condition. Let $\mathcal{G}_{E,D}$ denote the vector space of all $F \in \mathscr{S}_{E,D}$ satisfying

$$\forall_{\xi\in E,\, S\subset\subset\widehat{D}}\, \exists_{C_1,C_2>0}\, \forall_{n\in\mathbb{N}_0,\chi_1,\ldots,\chi_n\in S}\, \|F_n(\xi\otimes\chi_1\otimes\cdots\otimes\chi_n)\| \le C_1 C_2^n, \tag{2.4.7}$$

and define the subspace

$$\mathcal{G}_{E,D}^{\dagger} := \left\{F \in \mathscr{S}_{E,D}^{\dagger} \,\middle|\, F, F^{\dagger} \in \mathcal{G}_{E,D}\right\}, \tag{2.4.8}$$

again abbreviating to $\mathcal{G}_D$, $\mathcal{G}_D^{\dagger}$ when $E = \mathbb{C}$. The estimate (2.4.6) implies that if $F \in \mathcal{G}_{E,D}$, $\xi \in E$, $f \in \mathbb{S}_D$, then $\sum_{n\ge 0}\Lambda_t^n(F_n)\xi\varepsilon(f)$ is absolutely convergent, and the convergence is locally uniform in t. The resulting map

$$\Lambda : \mathcal{G}_{E,D} \to \mathbb{P}_{\mathrm{c}}(E, \mathcal{E}_D) \tag{2.4.9}$$

is linear. In the Guichardet notation (cf. Section 2.3) (2.4.5) yields the identity

$$\langle \xi\varepsilon(f), \Lambda_t(F)\eta\varepsilon(g)\rangle = \int_{\Gamma_{[0,t[}} d\sigma \left\langle \xi\pi_{\hat{f}}(\sigma), F_{\#\sigma}\eta\pi_{\hat{g}}(\sigma)\right\rangle \langle\varepsilon(f), \varepsilon(g)\rangle, \tag{2.4.10}$$

where $\xi, \eta \in E$, $f, g \in \mathbb{S}_D$ and $\int d\sigma$ denotes integration with respect to the Guichardet measure. Another application of (2.4.6) shows that indeed $\Lambda(\mathcal{G}_D) \subset \mathbb{P}_{\mathrm{Hc}}(\mathcal{E}_D)$. Moreover (2.4.2) implies that $\Lambda(\mathcal{G}_D^\dagger) \subset \mathbb{P}_{\mathrm{Hc}}^\dagger(\mathcal{E}_D)$ and for $F \in \mathcal{G}_{E,D}^\dagger$

$$\Lambda(F)^\dagger = \Lambda(F^\dagger). \tag{2.4.11}$$

A minor modification of the proof of Proposition 2.3 of [LW$_4$] yields injectivity of the map Λ.

The formulation of the Itô Formula for iterated QS integrals is postponed to respectively Sections 3.5 and 4.6 - it is more convenient to express it separately in the purely algebraic and in the operator space theoretic context.

Chapter 3

Algebraic case

This chapter is concerned with quantum stochastic convolution cocycles on a coalgebra. They are a natural, still purely algebraic, generalisation of quantum Lévy processes on a *-bialgebra defined by L. Accardi, W. von Waldenfels and M.Schürmann in [ASW].

QS convolution cocycles are obtained by solving coalgebraic QS differential equations. The original proof of the existence of solutions for such equations (for a particular type of the generator, guaranteeing that the solution is unital and *-homomorphic), due to M. Schürmann ([Sch]), exploited the formulation of QS integrals in terms of integral-sum kernels. Here the proof is simplified by using the language of iterated QS integration of matrix-sum kernels, whose use has been promoted in [LW_3] and in [LW_4]. The uniqueness of weak solutions is also proved. The basic tool is the Fundamental Theorem on Coalgebras, allowing essentially to reduce the considered problem to the finite dimensional case.

The cocycles arising in this way satisfy a Hölder condition, and it is shown that conversely every such Hölder-continuous cocycle is governed by a QS differential equation. Algebraic structure enjoyed by matrix-sum kernels yields a unital *-algebra of processes which allows easy deduction of homomorphic properties of cocycles on a *-bialgebra, the stochastic generators of which can be described in terms of so-called Schürmann triples. This in particular yields a simple proof of the Schürmann Reconstruction Theorem - every quantum Lévy process may be equivalently realised in Fock space. The perturbation of QS convolution cocycles by Weyl cocycles is shown to be implemented on the level of their stochastic generators by the action of the corresponding Euclidean group on Schürmann triples.

There is also a corresponding concept of opposite QS convolution cocycles that can be obtained from the usual ones either by the time-inversion or by using the opposite comultiplication on a coalgebra. Opposite cocycles also satisfy QS differential equations.

3.1 Coalgebras and convolution semigroups

In this section we present the definition of a coalgebra and establish basic facts concerning the convolution product provided by the coalgebraic structure.

Definition 3.1.1. A vector space $\mathcal{C}$ is a coalgebra if there are linear maps $\Delta : \mathcal{C} \to \mathcal{C} \odot \mathcal{C}$ and $\epsilon : \mathcal{C} \to \mathbb{C}$, called the *coproduct* and *counit* respectively, enjoying the coassociativity and the counit property, namely

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta, \tag{3.1.1}$$

$$(\mathrm{id} \otimes \epsilon) \circ \Delta = (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}. \tag{3.1.2}$$

The main examples of coalgebras of interest derive from the theory of quantum groups. Sometimes it will be handy to think of $\mathbb{C}$ as a coalgebra, with both coproduct and counit given by the identity mapping.

M.E. Sweedler has introduced the notation $a_{(1)} \otimes a_{(2)}$ for Δa, in which both summation and indices are supressed ([Swe]). With this, (3.1.1) and (3.1.2) read

$$a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)} = a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)}, \quad \text{and} \quad a_{(1)}\epsilon(a_{(2)}) = \epsilon(a_{(1)})a_{(2)} = a.$$

Let $\Delta_0 := \mathrm{id}_{\mathcal{C}}$ and for $n \in \mathbb{N}$ define

$$\Delta_n := \big(\mathrm{id}^{\otimes(n-1)} \otimes \Delta\big) \circ \cdots \circ (\mathrm{id} \otimes \Delta) \circ \Delta. \tag{3.1.3}$$

The coassociativity implies that moving any Δ to any of the available tensor places within its bracket (rather than the right-most, as here) has no effect. It is easily verified that the family $\{\Delta_n : n \in \mathbb{N}_0\}$ satisfies

$$\big(\Delta_i \otimes \Delta_j\big) \circ \Delta = \Delta_{i+j+1}. \tag{3.1.4}$$

The Sweedler notation extends to writing $a_{(1)} \otimes \cdots \otimes a_{(n+1)}$ for $\Delta_n a$ ($n \geq 1$), $a \in \mathcal{C}$. Thus, for example, $a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$ becomes a neutral notation for the effect of (3.1.1) on an element a.

The following important result holds (the proof may be found for example in [Sch]):

Theorem 3.1.2 (Fundamental Theorem on Coalgebras). *Let $\mathcal{C}$ be a coalgebra, $a \in \mathcal{C}$. The subcoalgebra of $\mathcal{C}$ generated by a is finite-dimensional.*

Let U, V, W be vector spaces and assume there is a natural map $\cdot : (U \odot V) \to W$ (for example $V = \mathbb{C}$, $U = W$). For linear maps $\alpha : \mathcal{C} \to U$, $\beta : \mathcal{C} \to V$ define

$$\alpha \star \beta := \cdot(\alpha \otimes \beta)\Delta : \mathcal{C} \to W. \tag{3.1.5}$$

Thus, for example, the counit property (3.1.2) implies that $\epsilon \star \alpha = \alpha \star \epsilon = \alpha$ for any linear map α from $\mathcal{C}$ into a vector space. In particular $\big(L(\mathcal{C}; \mathbb{C}), \star\big)$ is a unital algebra with identity ϵ.

A functional $\lambda : \mathcal{C} \to \mathbb{C}$ is called *idempotent* if $\lambda = \lambda \star \lambda$ (so for example the counit is idempotent).

Definition 3.1.3. A convolution semigroup of functionals (CSF, for short) on a coalgebra $\mathcal{C}$ is a family $\{\kappa_s : s \geq 0\}$ of linear functionals on $\mathcal{C}$ satisfying the following conditions::

$$\kappa_{s+t} = \kappa_s \star \kappa_t, \quad s, t \geq 0, \tag{3.1.6}$$

$$\kappa_0 = \epsilon. \tag{3.1.7}$$

If for each $a \in \mathcal{C}$

$$\lim_{t \to 0^+} \kappa_t(a) = \epsilon(a)$$

then $\{\kappa_s : s \geq 0\}$ is said to be continuous (CCSF).

The following fact appeared first in [ASW], for another proof see [Sch]. It is a straightforward consequence of the Fundamental Theorem on Coalgebras.

Fact 3.1.4. *Each CCSF* $\{\kappa_s : s \geq 0\}$ *has a* generator $\gamma \in L(\mathcal{C}; \mathbb{C})$ *given by*

$$\gamma(a) = \lim_{t \to 0^+} \frac{\kappa_t(a) - \epsilon(a)}{t}, \quad a \in \mathcal{C}.$$

The CCSF may be recovered from its generator by the formula

$$\kappa_t(a) = \exp_\star t\gamma(a) := \sum_{n \geq 0} (n!)^{-1} t^n \gamma^{\star n}(a) \tag{3.1.8}$$

($a \in \mathcal{C}$) where $\gamma^{\star 0} := \epsilon$.

Definition 3.1.5. A coalgebra $\mathcal{C}$ is involutive if there exists a conjugate linear involution * compatible with the coalgebra operations: $\epsilon(a^*) = \overline{\epsilon(a)}$, $\Delta(a^*) = (a_{(1)})^* \otimes (a_{(2)})^*$. An algebra $\mathcal{A}$ is a bialgebra if it is also a coalgebra with multiplicative coproduct and counit. It is unital if it is unital as an algebra and $\epsilon(1) = 1$, $\Delta(1) = 1 \otimes 1$. Finally a *-bialgebra is a bialgebra with involution which is both algebraic and coalgebraic.

An important example of an idempotent functional on a unital *-bialgebra $\mathcal{A}$ is so-called *Haar state*, i.e. a state (positive unital functional) $h : \mathcal{A} \to \mathbb{C}$ satisfying the (left and right) invariance properties:

$$h(a_{(1)})a_{(2)} = a_{(1)}h(a_{(2)}) = h(a)1 \tag{3.1.9}$$

for all $a \in \mathcal{A}$. It is easy to see that the Haar state on $\mathcal{A}$ is unique (if it exists).

Note the following traffic between properties of a CCSF $(\kappa_t)_{t \geq 0}$ and its generator γ when the coalgebra has more structure. The semigroup consists of real (respectively, unital) functionals if and only if the generator is real (resp. vanishes at the unit). Moreover, on a *-bialgebra $\mathcal{A}$, if the functionals are positive then the generator is *conditionally positive*:

$$\gamma(a) \geq 0 \text{ for } a \in \mathcal{A}_+ \cap \operatorname{Ker} \epsilon, \tag{3.1.10}$$

as for all $a \in \mathcal{A}_+ \cap \operatorname{Ker} \epsilon$, $t \geq 0$,

$$\frac{\kappa_t(a) - \epsilon(a)}{t} = \frac{\kappa_t(a)}{t} \geq 0.$$

3.2 QS convolution cocycles

Here we introduce for the first time the main object analysed in the thesis.

Definition 3.2.1. A quantum stochastic convolution cocycle (on $\mathcal{C}$ with domain $\mathcal{E}_D$) is a process $l \in \mathbb{P}(\mathcal{C}; \mathcal{E}_D)$ such that, for all $s, t \geq 0$,

$$l_{s+t} = l_s \star (\sigma_s \circ l_t) \tag{3.2.1}$$

and for all $a \in \mathcal{C}$

$$l_0(a) = \epsilon(a) I_{\mathcal{E}_D}. \tag{3.2.2}$$

Family of all QS convolution cocycles is denoted by $\mathbb{QSCC}(\mathcal{C}; \mathcal{E}_D)$; the notation adorned with subscripts and superscript † according to our convention.

Remark 3.2.2. The formula (3.2.1) makes use of the assumed adaptedness of l and the identification $\mathcal{F}_{[0,s+t)} \cong \mathcal{F}_{[0,s)} \otimes \mathcal{F}_{[s,s+t)}$. It is referred to as the *convolution increment property*, and any process satisfying it is called a *convolution increment process*. The initial condition for a convolution increment process $k \in \mathbb{P}(\mathcal{C}; \mathcal{E}_D)$ must have the form $k_0(a) = \lambda(a) I_{\mathcal{E}_D}$ $(a \in \mathcal{C})$, where $\lambda : \mathcal{C} \to \mathbb{C}$ is an idempotent functional.

The convolution increment property has an equivalent 'weak' description:

Lemma 3.2.3. *Let $l \in \mathbb{P}(\mathcal{C}; \mathcal{E}_D)$. Then l is a convolution increment process if and only if for each $f, g \in \mathbb{S}_D$,*

$$\mathrm{e}^{-\langle f_{[0,t)}, g_{[0,t)} \rangle} \left\langle \varepsilon(f_{[0,t)}), l_t(a) \varepsilon(g_{[0,t)}) \right\rangle = \prod_{i=0}^{n-1} \lambda_{t_{i+1}-t_i}^{f(t_i), g(t_i)}(a_{(i+1)}) \tag{3.2.3}$$

where $0 = t_0 \leq t_1 \leq \cdots \leq t_n = t$ *contains the discontinuities of* $f_{[0,t)}$ *and* $g_{[0,t)}$*,* $a_{(1)} \otimes \cdots \otimes a_{(n)} = \Delta_{n-1}(a)$ *(Sweedler-style), and*

$$\lambda_t^{c,d} := \mathrm{e}^{-t\langle c,d\rangle} \left\langle \varepsilon(c_{[0,t[}), l_t(\,\cdot\,)\varepsilon(d_{[0,t[}) \right\rangle . \tag{3.2.4}$$

Proof. The identity (3.2.3) results from repeated application of the cocycle relation (3.2.1). The other direction is trivial. □

Corollary 3.2.4. *Let* $l \in \mathbb{QSCC}_{\mathrm{wc}}(\mathcal{C}; \mathcal{E}_D)$*,* $c, d \in D$*. Then* (3.2.4) *defines a continuous convolution semigroup of functionals on* $\mathcal{C}$ *:* $(\lambda_t^{c,d})_{t\geq 0}$*.*

Definition 3.2.5. Semigroups defined by (3.2.4) (for all pairs $c, d \in D$) are called *associated convolution semigroups* of the cocycle l.

It is clear that two QS convolution cocycles with identical convolution semigroups are equal. The above 'semigroup decomposition' of a QS convolution cocycle is crucial for the results in Section 3.4.

Observe that if $\mathcal{C}$ is a trivial coalgebra $\mathbb{C}$, the QS convolution cocycle l on $\mathbb{C}$ is in fact determined by the operator process $l(1)$, and the latter is an *operator Markovian cocycle*:

Definition 3.2.6. A process $X \in \mathbb{P}(\mathcal{E}_D)$ is called an operator Markovian cocycle if $X_0 = I_{\mathcal{E}_D}$ and

$$X_{s+t} = X_s \sigma_s(X_t), \quad s, t \geq 0.$$

3.3 Coalgebraic QS differential equations

In this section we describe the conditions assuring the existence and uniqueness of solutions of coalgebraic QS differential equations and present basic properties of the solutions.

Let $\varphi \in L\big(\mathcal{C}; \mathcal{O}(\widehat{D})\big)$. A *coalgebraic QS differential equation* (with the coefficient φ) is the equation of the form

$$dk_t = k_t \star_\tau \varphi \, d\Lambda_t, \quad k_0 = \iota \circ \epsilon \tag{3.3.1}$$

(τ denoting a tensor flip exchanging the order of $\widehat{\mathsf{k}}$ and $\mathcal{F}$, ι indicating an ampliation). A process $k \in \mathbb{P}(\mathcal{C}, \mathcal{E}_D)$ is a *weak solution* of the equation (3.3.1) if for all $f, g \in \mathbb{S}_D$, $t \geq 0, a \in \mathcal{C}$

(3.3.2)
$$\Big\langle \varepsilon(f), \big(k_t(a) - \epsilon(a) I_{\mathcal{F}}\big)\varepsilon(g)\Big\rangle = \int_0^t \Big\langle \hat{f}(s), \varphi(a_{(2)})\hat{g}(s)\Big\rangle \big\langle \varepsilon(f), k_s(a_{(1)})\varepsilon(g)\big\rangle \, ds.$$

Thus a weak solution k is necessarily weakly continuous, and therefore also weakly regular by (2.3.4). If $k \in \mathbb{P}_2(\mathcal{C}; \mathcal{E}_D)$ then the First Fundamental Formula implies that
$$k_t(a) = \epsilon(a) I + \int_0^t \varphi(a_{(2)}) \otimes k_s(a_{(1)}) \, d\Lambda_s,$$
and k is called a *strong solution* of (3.3.1).

Uniqueness of the solution

Before establishing the uniqueness of the solution of (3.3.1) we need a following lemma:

Lemma 3.3.1. *Let $k \in \mathbb{P}_{\mathrm{wr}}(\mathcal{C}; \mathcal{E}_D)$ and let V be a finite dimensional subspace of $\mathcal{C}$ equipped with some norm $\|\cdot\|$. Then, for each $f, g \in \mathbb{S}_D$ and $T \geq 0$,*

(3.3.3)
$$C_{f,g,T,V} := \sup\Big\{ \big| \langle \varepsilon(f), k_t(a)\varepsilon(g)\rangle \big| \, : \, a \in V, \|a\| \leq 1, \, 0 \leq t \leq T \Big\} < \infty.$$

Proof. Let $e_1, \dots e_N$ be a basis for V and $\|\cdot\|'$ be the l^1-norm with respect to this basis. Then, for $a \in V$,
$$\big| \langle \varepsilon(f), k_t(a)\varepsilon(g)\rangle \big| \leq \|a\|' \max_i \big| \langle \varepsilon(f), k_t(e_i)\varepsilon(g)\rangle \big|.$$
Since all norms on V are equivalent, the result follows. □

Define $\phi : \mathcal{C} \to \mathcal{C} \odot \mathcal{O}(\widehat{D})$ by

$$\phi = (\mathrm{id}_{\mathcal{C}} \otimes \varphi) \circ \Delta, \tag{3.3.4}$$

Proposition 3.3.2. *The coalgebraic QS differential equation* (3.3.1) *has at most one weak solution.*

Proof. Let $k \in \mathbb{P}(\mathcal{C}; \mathcal{E}_D)$ be the difference of two weak solutions of (3.3.1), and let $a \in \mathcal{C}$, $f, g \in \mathbb{S}_D$ and $T \geq 0$. Then $k \in \mathbb{P}_{\mathrm{wc}}(\mathcal{C}; \mathcal{E}_D) \subset \mathbb{P}_{\mathrm{wr}}(\mathcal{C}; \mathcal{E}_D)$, by (2.3.4). By iteration

$$\langle \varepsilon(f), k_t(a)\varepsilon(g) \rangle = \int_{\Delta_n[0,t]} \Big\langle \varepsilon(f), k_{s_1}\Big(\phi^{\hat{f}(s_1)}_{\hat{g}(s_1)} \circ \cdots \circ \phi^{\hat{f}(s_n)}_{\hat{g}(s_n)}(a)\Big)\varepsilon(g) \Big\rangle \, d\boldsymbol{s}$$

for each $n \in \mathbb{N}$ and $t \in [0, T]$, where for $\xi, \eta \in \widehat{D}$

$$\phi^{\xi}_{\eta} := \big(\mathrm{id}_{\mathcal{C}} \otimes (\omega_{\xi,\eta} \circ \varphi)\big) \circ \Delta$$

(recall the notation (2.1.2)). The coalgebra $\mathcal{C}_a$, defined as the subcoalgebra of $\mathcal{C}$ generated by a is finite-dimensional by Theorem 3.1.2. As each ϕ^{ξ}_{η} leaves $\mathcal{C}_a$ invariant, fixing a norm for $\mathcal{C}_a$ and appealing to Lemma 3.3.1 allows us to claim that the integrand is bounded by

$$\Big(\max\big\{\|\phi^{\hat{f}(s)}_{\hat{g}(s)}\| : 0 \leq s \leq T\big\}\Big)^n C_{f,g,T,\mathcal{C}_a}\|a\|.$$

The result follows. □

Existence of the solution

Let υ^{φ} be the linear map $\mathcal{C} \to \mathscr{S}_D$ defined by $\upsilon^{\varphi}(a)_n = \upsilon^{\varphi}_n(a)$ ($n \in \mathbb{N}_0$) where $\upsilon^{\varphi}_0 := \epsilon$ and, in the notation (3.1.3),

$$\upsilon^{\varphi}_n = \varphi^{\otimes n} \circ \Delta_{n-1} : \mathcal{C} \to \mathcal{O}(\widehat{D})^{\odot n} \subset \mathcal{O}(\widehat{D}^{\odot n}) \text{ for } n \geq 1. \tag{3.3.5}$$

Note the recursive identity

$$\upsilon^{\varphi}_{n+1} = (\upsilon^{\varphi}_n \otimes \varphi) \circ \Delta. \tag{3.3.6}$$

If $\mathcal{C}$ is involutive and $\varphi \in L(\mathcal{C}; \mathcal{O}^\dagger(\widehat{D}))$ then

$$(\upsilon^\varphi)^\dagger = \upsilon^{\varphi^\dagger}. \tag{3.3.7}$$

In terms of the associated map ϕ introduced in (3.3.4)

$$\upsilon^\varphi_n = \epsilon_n \circ \phi_n,$$

where the maps $\phi_n : \mathcal{C} \to \mathcal{C} \odot \mathcal{O}(\widehat{D})^{\odot n}$ and $\epsilon_n : \mathcal{C} \odot \mathcal{O}(\widehat{D})^{\odot n} \to \mathcal{O}(\widehat{D})^{\odot n}$ are defined (recursively) by $\phi_0 = \mathrm{id}_{\mathcal{C}}$, $\epsilon_0 = \epsilon$ and, for $n \geq 1$, $\phi_n = (\phi \otimes \mathrm{id}_{\mathcal{O}(\widehat{D})^{\odot n}}) \circ \phi_{n-1}$, $\epsilon_n = \epsilon \otimes \mathrm{id}_{\mathcal{O}(\widehat{D})^{\odot n}}$.

Define additionally $\widetilde{\upsilon}^\varphi : \mathcal{C} \to \mathscr{S}_D$ by

$$\widetilde{\upsilon}^\varphi_n = \tau_n \circ \upsilon^\varphi_n, \quad n \in \mathbb{N}_0, \tag{3.3.8}$$

where $\tau_n : \mathcal{O}(\widehat{D}^{\odot n}) \to \mathcal{O}(\widehat{D}^{\odot n})$ denotes the flip reversing the order of the n copies of $\widehat{D}$.

Lemma 3.3.3. *For any $\varphi \in L(\mathcal{C}; \mathcal{O}(\widehat{D}))$, $\upsilon^\varphi \in L(\mathcal{C}; \mathcal{G}_D)$ and $\widetilde{\upsilon}^\varphi \in L(\mathcal{C}; \mathcal{G}_D)$. Moreover, if φ is $\mathcal{O}^\dagger(\widehat{D})$-valued then both υ^φ and $\widetilde{\upsilon}^\varphi$ take values in $\mathcal{G}^\dagger_D$.*

Proof. Fix an element $a \in \mathcal{C} \setminus \{0\}$ and let again $\mathcal{C}_a$ denote the coalgebra generated by a. By Theorem 3.1.2 $\mathcal{C}_a$ is finite dimensional; let $a^1, \ldots, a^N$ be its linear basis in which $a^1 = a$. Let (ν^i_{jk}) be the coefficients of Δ (viewed as a map $\mathcal{C}_a \to \mathcal{C}_a \otimes \mathcal{C}_a$) with respect to this basis, i.e. for all $i = 1, \ldots, N$

$$\Delta(a^i) = \sum_{j,k=1}^N \nu^i_{j,k} a^j \otimes a^k.$$

Set $(i, j = 1, \ldots, N)$

$$T^i_j = \sum_k \nu^i_{jk} \varphi(a^k) \in \mathcal{O}(\widehat{D}).$$

Then $\phi(a^i) = \sum_{j=1}^N a^j \otimes T^i_j$ and

$$\upsilon^\varphi_n(a) = \sum_{\mathbf{k}} \epsilon(a^{k_n}) T^{k_{n-1}}_{k_n} \otimes T^{k_{n-2}}_{k_{n-1}} \otimes \cdots \otimes T^1_{k_1}, \tag{3.3.9}$$

a sum of N^n terms of the form $X_1 \otimes \cdots \otimes X_n$ in which

$$X_i \in \{T^j_k : 1 \le j, k \le N\} \cup \{\epsilon(a^j)T^l_j : 1 \le j, l \le N\},$$

so $\upsilon^\varphi(a) \in \mathcal{G}_D$. The rest is easily verified. □

Theorem 3.3.4. *Let $\varphi \in L\left(\mathcal{C}; \mathcal{O}(\widehat{D})\right)$ and set $\widetilde{\upsilon} = \widetilde{\upsilon}^\varphi$. Then the process $k := \Lambda \circ \widetilde{\upsilon}$ strongly satisfies the coalgebraic quantum stochastic differential equation* (3.3.1)*.*

Proof. By (2.4.9) and Lemma 3.3.3 the process k is continuous. It therefore suffices to show that it satisfies the equation weakly. In Guichardet notation, (2.4.10), (3.3.6) and (3.3.8) imply

$$\begin{aligned}
&\int_0^t ds \left\langle \hat{f}(s), \varphi(a_{(2)})\hat{g}(s) \right\rangle \left\langle \varepsilon(f), k_s(a_{(1)})\varepsilon(g) \right\rangle \\
&= \int_0^t ds \int_{\Gamma_{[0,s]}} d\tau \left\langle \pi_{\hat{f}}(\tau \cup s), \big(\varphi(a_{(2)}) \otimes \widetilde{\upsilon}_{\#\tau}(a_{(1)})\big)\pi_{\hat{g}}(\tau \cup s) \right\rangle \langle \varepsilon(f), \varepsilon(g) \rangle \\
&= \int_{\Gamma_{[0,t]}} d\sigma \big(1 - \delta_\emptyset(\sigma)\big) \left\langle \pi_{\hat{f}}(\sigma), \widetilde{\upsilon}_{\#\sigma}(a)\pi_{\hat{g}}(\sigma) \right\rangle \langle \varepsilon(f), \varepsilon(g) \rangle \\
&= \langle \varepsilon(f), k_t(a)\varepsilon(g) \rangle - \epsilon(a) \langle \varepsilon(f), \varepsilon(g) \rangle,
\end{aligned}$$

and so, by (3.3.2), k is the weak solution of (3.3.1). □

Thus the coalgebra QS differential equation (3.3.1) has a unique weak solution; it is a strong solution and is given by $\Lambda \circ \widetilde{\upsilon}^\varphi$ — we denote it l^φ.

Properties of the solution

Lemma 3.3.5. *Let $\varphi \in L\left(\mathcal{C}; \mathcal{O}(\widehat{D})\right)$. Then $l^\varphi \in \mathbb{P}_{\mathrm{Hc}}(\mathcal{C}; \mathcal{E}_D)$ and the following hold.*

(a) *The map* $\varphi \mapsto l^\varphi$ *is injective.*

(b) *If* $\varphi \in L\big(\mathcal{C}; \mathcal{O}^\dagger(D)\big)$ *then* $l^\varphi \in \mathbb{P}^\dagger_{\mathrm{Hc}}(\mathcal{C}; \mathcal{E}_D)$.

(c) *If* $\mathcal{C}$ *is unital then* l^φ *is unital if and only if* $\varphi(1) = 0$.

(d) *If* $\mathcal{C}$ *is involutive and* $\varphi \in L\Big(\mathcal{C}; \mathcal{O}^\dagger(\widehat{D})\Big)$ *then* $(l^\varphi)^\dagger = l^{\varphi^\dagger}$. *In particular,* l^φ *is real if and only if* φ *is real.*

Proof. In view of the inclusion $\Lambda(\mathcal{G}_D) \subset \mathbb{P}_{\mathrm{Hc}}(\mathcal{E}_D)$ and Lemma 3.3.3, l^φ belongs to $\mathbb{P}_{\mathrm{Hc}}(\mathcal{C}; \mathcal{E}_D)$.
(a) follows from the identity

$$\Big\langle \hat{c}, \varphi(a)\hat{d} \Big\rangle = \lim_{t \to 0^+} t^{-1}\Big(\big\langle \varepsilon(c_{[0,t)}), l^\varphi_t(a)\varepsilon(d_{[0,t)})\big\rangle - \epsilon(a)\mathrm{e}^{t\langle c,d\rangle}\Big), \tag{3.3.10}$$

and the totality of $\{\hat{c} : c \in D\}$ in $\widehat{\mathsf{k}}$.
(b) follows from Lemma 3.3.3 and (2.4.11).
(c) follows from (3.3.2), (3.3.10) and the unitality of ϵ and Δ.
(d) By part (b) $l^\varphi \in \mathbb{P}^\dagger_{\mathrm{Hc}}(\mathcal{C}, \mathcal{E}_D)$ and by (3.3.7) and (2.4.11)

$$(l^\varphi)^\dagger = \big(\Lambda \circ (\widetilde{\upsilon}^\varphi)\big)^\dagger = \Lambda \circ \widetilde{\upsilon}^{\varphi^\dagger} = l^{\varphi^\dagger}.$$

The last part follows by injectivity. □

Remark 3.3.6. If we cast $\varphi \in L\big(\mathcal{C}; \mathcal{O}(\widehat{D})\big)$ in block matrix form:

$$\varphi = \begin{bmatrix} \gamma & \alpha \\ \chi & \nu - \iota \circ \epsilon \end{bmatrix},$$

where $\iota(z) = zI_D$, then $\gamma \in L(\mathcal{C}; \mathbb{C})$, $\chi \in L\big(\mathcal{C}; L(\mathbb{C}; D)\big)$, $\alpha \in L\big(\mathcal{C}; L(D; \mathbb{C})\big)$ and $\nu \in L\big(\mathcal{C}; \mathcal{O}(D)\big)$, so that

$$\varphi(a) = \begin{bmatrix} \gamma(a) & \alpha(a) \\ \chi(a) & \nu(a) - \epsilon(a)I \end{bmatrix}. \tag{3.3.11}$$

Moreover $\varphi \in L\big(\mathcal{C}; \mathcal{O}^\dagger(\widehat{D})\big)$ if and only if $\alpha(\mathcal{C}) \subset \big\langle D\big| := \big\{\langle d| \,\big|\, d \in D\big\}$ and $\nu(\mathcal{C}) \subset \mathcal{O}^\dagger(D)$. Thus if $\mathcal{C}$ is involutive then

$$\varphi^\dagger = \begin{bmatrix} \gamma^\dagger & \chi^\dagger \\ \alpha^\dagger & \nu^\dagger - \iota \circ \epsilon \end{bmatrix}.$$

In particular,

$$\varphi = \varphi^\dagger \text{ if and only if } \gamma = \gamma^\dagger,\ \nu = \nu^\dagger \text{ and } \alpha = \chi^\dagger.$$

3.4 Stochastically generated convolution cocycles

In this section we first note that solutions of coalgebraic QS differential equations are QS convolution cocycles. Heeding the fact that solutions are $\frac{1}{2}$-Hölder continuous we then establish the converse: every QS convolution cocycle in $\mathbb{P}^\dagger_{\mathrm{Hc}}(\mathcal{C}; \mathcal{E}_D)$ necessarily satisfies a coalgebraic QS differential equation.

Lemma 3.4.1. *Let $\varphi \in L(\mathcal{C}; \mathcal{O}(\widehat{D}))$. Then $\upsilon := \upsilon^\varphi$ defined in* (3.3.5) *satisfies*

$$\upsilon_{n+m} = (\upsilon_n \otimes \upsilon_m) \circ \Delta$$

for all $n, m \in \mathbb{N}_0$.

Proof. Since $\upsilon_k = \varphi^{\otimes k} \circ \Delta_{k-1}$, this reduces to the identity (3.1.4). $\square$

Proposition 3.4.2. *Let $l = l^\varphi$ where $\varphi \in L(\mathcal{C}; \mathcal{O}(\widehat{D}))$. Then $l \in \mathbb{QSCC}_{\mathrm{Hc}}(\mathcal{C}; \mathcal{E}_D)$. If $\mathcal{C}$ is involutive and φ takes values in $\mathcal{O}^\dagger(\widehat{D})$ then $l \in \mathbb{QSCC}^\dagger_{\mathrm{Hc}}(\mathcal{C}; \mathcal{E}_D)$*

Proof. Set $\widetilde{\upsilon} = \widetilde{\upsilon}^\varphi$ as in Theorem 3.3.4 and let $a \in \mathcal{C}$. Using Guichardet

notation again, we obtain for all $f, g \in \mathbb{S}_D$, $s, t \geq 0$

$$
\begin{aligned}
&\langle \varepsilon(f), (l_s \star (\sigma_s \circ l_t))(a)\varepsilon(g)\rangle = \big\langle \varepsilon(f), \big(l_s(a_{(1)}) \otimes (\sigma_s \circ l_t)(a_{(2)})\big)\,\varepsilon(g)\big\rangle \\
&= \big\langle \varepsilon(f_{[0,s)}), l_s(a_{(1)})\varepsilon(g_{[0,s)})\big\rangle \big\langle \varepsilon(f_{[s,\infty)}), \sigma_s(l_t(a_{(2)}))\varepsilon(g_{[s,\infty)})\big\rangle \\
&= \int_{\Gamma_{[0,s[}} d\sigma \left\langle \pi_{\hat{f}}(\sigma), \widetilde{\upsilon}_{\#\sigma}(a_{(1)})\pi_{\hat{g}}(\sigma)\right\rangle \int_{\Gamma_{[s,s+t[}} d\tau \left\langle \pi_{\hat{f}}(\tau), \widetilde{\upsilon}_{\#\tau}(a_{(2)})\pi_{\hat{g}}(\tau)\right\rangle \mathrm{e}^{\langle f,g\rangle} \\
&= \int_{\Gamma_{[0,s+t[}} d\omega \left\langle \pi_{\hat{f}}(\omega), \widetilde{\upsilon}_{\#\omega\cap[s,s+t)}(a_{(2)}) \otimes \widetilde{\upsilon}_{\#\omega\cap[0,s)}(a_{(1)})\pi_{\hat{g}}(\omega)\right\rangle \mathrm{e}^{\langle f,g\rangle} \\
&= \int_{\Gamma_{[0,s+t[}} d\omega \left\langle \pi_{\hat{f}}(\omega), \tau_{\#\omega}\left(\upsilon_{\#\omega\cap[0,s)}(a_{(1)}) \otimes \upsilon_{\#\omega\cap[s,s+t)}(a_{(2)})\right) \pi_{\hat{g}}(\omega)\right\rangle \mathrm{e}^{\langle f,g\rangle}.
\end{aligned}
$$

Applying Lemma 3.4.1, and identity (2.4.10) once more, we see that this is equal to $\langle \varepsilon(f), l_{s+t}(a)\varepsilon(g)\rangle$, which proves that l is a convolution increment process. As its initial condition is given by the counit, Lemma 3.3.5 ends the proof. $\square$

Thus $\varphi \mapsto l^\varphi$ gives maps

$$L\big(\mathcal{C}; \mathcal{O}(\widehat{D})\big) \to \mathbb{QSCC}_{\mathrm{Hc}}(\mathcal{C}; \mathcal{E}_D) \text{ and } L\big(\mathcal{C}; \mathcal{O}^\dagger(\widehat{D})\big) \to \mathbb{QSCC}^\dagger_{\mathrm{Hc}}(\mathcal{C}; \mathcal{E}_D).$$

These maps are injective, by Lemma 3.3.5; our aim now is to establish bijectivity of the second map. When $l = l^\varphi$ we refer to φ as the *stochastic generator* of the QS convolution cocycle l. Note that if l is a cocycle in $\mathbb{P}^\dagger(\mathcal{C}; \mathcal{E}_D)$, and $\mathcal{C}$ is involutive, then $l^\dagger$ is a cocycle too.

Recall Definition 3.2.5 and Corollary 3.2.4.

Lemma 3.4.3. *Let $l = l^\varphi$ for $\varphi \in L\big(\mathcal{C}; \mathcal{O}(\widehat{D})\big)$. Then its associated CCSFs $\lambda^{c,d}$ have generators $\langle \hat{c}, \varphi(\,\cdot\,)\hat{d}\rangle$ $(c, d \in D)$.*

Proof. Since $l_t(a) = \Lambda_t\big(\widetilde{\upsilon}^\varphi(a)\big)$ this is an immediate consequence of (2.4.10) and the definition of υ^φ by formula (3.3.5) - see also the proof of Lemma 3.3.5. $\square$

Proposition 3.4.4. *Let $l \in \mathbb{QSCC}_{\mathrm{c}}^{\dagger}(\mathcal{C}; \mathcal{E}_D)$. Define a map*

$$q : \widehat{D} \times \widehat{D} \to L(\mathcal{C}; \mathbb{C}), \quad \left(\begin{pmatrix} z \\ c \end{pmatrix}, \begin{pmatrix} w \\ d \end{pmatrix} \right) \mapsto \begin{bmatrix} \overline{z-1} & 1 \end{bmatrix} \begin{bmatrix} \gamma_{0,0} & \gamma_{0,d} \\ \gamma_{c,0} & \gamma_{c,d} \end{bmatrix} \begin{bmatrix} w-1 \\ 1 \end{bmatrix},$$

where $\{\gamma_{c,d} : c, d \in D\}$ are the generators of the CCSFs associated with l. Then q is sesquilinear.

Proof. The proposition amounts to the sesquilinearity of each form $q_a := q(\cdot,\cdot)(a)$. Thus let $a \in \mathcal{C}$. First note the identity

$$q_a(\chi, \eta) = \lim_{t \to 0^+} t^{-1} \left\langle \xi(t), \left(w, d_{[0,t)}, (2!)^{-1/2} (d_{[0,t)})^{\otimes 2}, \cdots \right) \right\rangle,$$

for $\chi = \binom{z}{c}$ and $\eta = \binom{w}{d}$ in $\widehat{D}$, where

$$\xi(t) = \left[l_t(a)^\dagger - \overline{\epsilon(a)} I \right] \left((z-1)\varepsilon(0) + \varepsilon(c_{[0,t[}) \right).$$

Thus if $\eta = \eta_1 + \alpha\eta_2$, $\eta_1 = \binom{w_1}{d_1}$, $\eta_2 = \binom{w_2}{d_2}$ then

$$q_a(\chi, \eta) - q_a(\chi, \eta_1) - \alpha q_a(\chi, \eta_2) = \lim_{t \to 0^+} \langle \xi(t), \zeta(t) \rangle$$

where

$$\zeta(t) = t^{-1} \Big((n!)^{-1/2} \left[d^{\otimes n} - (d_1)^{\otimes n} - \alpha (d_2)^{\otimes n} \right] \otimes 1_{[0,t[^n} \Big)_{n \geq 2}.$$

Since ζ is locally bounded and $\xi(t) \to 0$ as $t \to 0^+$, by the continuity of the process $\left(l_t(a)^\dagger \right)_{t \geq 0}$, this shows that q_a is linear in its second argument. A very similar argument, this time using the continuity of the process $\left(l_t(a) \right)_{t \geq 0}$, shows that q_a is conjugate linear in its first argument. The result follows. □

Proposition 3.4.5. *Let $l \in \mathbb{QSCC}_{\mathrm{Hc}}^{\dagger}(\mathcal{C}; \mathcal{E}_D)$ and let q be defined as in* Proposition 3.4.4. *Then, for each $a \in \mathcal{C}$, the sesquilinear form $q(\cdot,\cdot)(a)$ is separately continuous in each argument.*

Proof. Let $\{\gamma_{c,d} : c, d \in D\}$ be the generators of CCSFs associated with l, let

$a \in \mathcal{C}$, and $\chi = \binom{z}{c}$, $\eta = \binom{w}{d} \in \widehat{D}$. Then

$$\begin{aligned} q(\chi, \eta)(a) =& \overline{z}\big((w-1)\gamma_{0,0}(a) + \gamma_{0,d}(a)\big) + \\ & (w-1)\big(\gamma_{c,0}(a) - \gamma_{0,0}(a)\big) + \big(\gamma_{c,d}(a) - \gamma_{0,d}(a)\big) \end{aligned}$$

and, for $e \in D$ and $T > 0$,

$$\begin{aligned} & \big|\gamma_{c,e}(a) - \gamma_{0,e}(a)\big| \\ &= \lim_{t \to 0^+} t^{-1} \Big| \Big\langle e^{-t\langle e,c \rangle} \varepsilon(c_{[0,t)}) - \varepsilon(0), \big(l_t(a) - \epsilon(a)\big)\varepsilon(e_{[0,t)}) \Big\rangle \Big| \\ &= \lim_{t \to 0^+} t^{-1} \Big| e^{-t\langle c,e \rangle} \Big\langle \varepsilon(c_{[0,t)}) - \varepsilon(0), \big(l_t(a) - \epsilon(a)\big)\varepsilon(e_{[0,t)}) \Big\rangle \Big| \\ &\leq \limsup_{t \to 0^+} t^{-1/2} \big\|\varepsilon(c_{[0,t)}) - \varepsilon(0)\big\| t^{-1/2} \Big\| \big[l_t(a) - \epsilon(a)\big] \varepsilon(e_{[0,T)}) \Big\| \mathrm{e}^{-\|e\|^2(T-t)/2} \\ &\leq \|c\| C(a, e, T) \end{aligned}$$

for some constant C depending only on a, e and T. Thus, setting $T = 1$,

$$\begin{aligned} \big|q(\chi, \eta)(a)\big| &\leq |z| \big|(w-1)\gamma_{0,0}(a) + \gamma_{0,d}(a)\big| + \|c\| \big(|w-1| C(a, 0, 1) + C(a, d, 1)\big) \\ &\leq M \|\chi\| \end{aligned}$$

for a constant M depending only on a and η. This establishes continuity in the first argument. Continuity in the second argument is proved by a very similar argument, this time using the Hölder-continuity of the process $\big(l_t(a)^\dagger\big)_{t \geq 0}$. □

Remark 3.4.6. Proposition 3.4.5 may be proved in the same way under the following weaker hypothesis: estimates of the type

$$\limsup_{t \to 0^+} t^{-1} \Big| \big\langle \varepsilon(c_{[0,t)}) - \varepsilon(0), l_t(a) \varepsilon(d_{[0,t)}) \big\rangle \Big| \leq M_{a,d} \|c\|,$$

for constants $M_{a,d}$ depending only on a and d should be satisfied by both l and $l^\dagger$.

We are ready for the main result of this section.

Theorem 3.4.7. *Let $k \in \mathbb{P}(\mathcal{C};\mathcal{E}_D)$. Then the following are equivalent :*

(i) $k \in \mathbb{QSCC}^{\dagger}_{\mathrm{Hc}}(\mathcal{C};\mathcal{E}_D)$;

(ii) $k = l^{\varphi}$ *for some* $\varphi \in L\big(\mathcal{C};\mathcal{O}^{\dagger}(\widehat{D})\big)$.

Proof. Let $k \in \mathbb{QSCC}^{\dagger}_{\mathrm{Hc}}(\mathcal{C};\mathcal{E}_D)$ and let $\{\gamma_{c,d} : c, d \in D\}$ denote the generators of CCSFs associated with k. By Propositions 3.4.4 and 3.4.5, there is a map $\varphi \in L\big(\mathcal{C};\mathcal{O}^{\dagger}(\widehat{D})\big)$ such that

$$\left\langle \begin{pmatrix} z \\ c \end{pmatrix}, \varphi(a) \begin{pmatrix} w \\ d \end{pmatrix} \right\rangle = \begin{bmatrix} \overline{z-1} & 1 \end{bmatrix} \begin{bmatrix} \gamma_{0,0}(a) & \gamma_{0,d}(a) \\ \gamma_{c,0}(a) & \gamma_{c,d}(a) \end{bmatrix} \begin{bmatrix} w-1 \\ 1 \end{bmatrix},$$

in particular,

$$\big\langle \hat{c}, \varphi(\,\cdot\,)\hat{d}\,\big\rangle = \gamma_{c,d}.$$

Thus, by Lemma 3.4.3, the QS convolution cocycles l^{φ} and k have the same CCSFs and so coincide. Thus (i) implies (ii). The converse has already been established in Proposition 3.4.2 and Lemma 3.3.5(c). □

Remark 3.4.8. The transformation between the family $\{\gamma_{c,d} : c, d \in D\}$ and φ is a familiar one in the analysis of stochastic cocycle generators (cf. [LW$_1$]).

As a special case of theorem 3.4.7 for $\mathcal{C} = \mathbb{C}$, we obtain the following result.

Corollary 3.4.9. *Let $X \in \mathbb{P}^{\dagger}_{\mathrm{Hc}}(\mathcal{E}_D)$. Then the following are equivalent*:

(i) *X is an operator Markovian cocycle;*

(ii) *X satisfies a QS differential equation of the form $dX_t = (L \otimes X_t)d\Lambda_t$, $X_0 = I$, for some $L \in \mathcal{O}^{\dagger}(\widehat{D})$.*

This type of cocycle is used in Section 3.7 for perturbing QS convolution cocycles on a general coalgebra.

3.5 Multiplicativity

The section is devoted to characterising the stochastic generators of weakly multiplicative (and *-homomorphic) QS cocycles on a *-bialgebra $\mathcal{A}$.

Matrix-sum kernels revisited

First we introduce certain further spaces of unbounded operators useful for formulating the Itô Formula for iterated QS integrals in the purely algebraic context. For a dense subspace E of a Hilbert space h let

$$\mathcal{O}^{\mathrm{inv}}(E) := \big\{T \in \mathcal{O}(E) \,:\, \operatorname{Ran} T \subset E\big\},$$
$$\mathcal{O}^{*}(E) := \big\{T \in \mathcal{O}^{\dagger}(E) \,:\, T, T^{\dagger} \in \mathcal{O}^{\mathrm{inv}}(E)\big\}$$

("inv" for invariant).

Operator composition $\mathcal{O}^{\dagger}(E) \times \mathcal{O}^{\mathrm{inv}}(E) \to \mathcal{O}(E)$ extends to pairs (S, T) in $\mathcal{O}^{\dagger}(E) \times \mathcal{O}(E)$ for which $\operatorname{Dom}(S^{\dagger})^{*} \supset \operatorname{Ran} T$, as follows:

$$S \cdot T := (S^{\dagger})^{*} T. \tag{3.5.1}$$

This partially defined product is bilinear in an obvious sense. Associativity relations however have to be justified (if needed) in every considered case separately.

The definitions above enable us to introduce the following subspaces of $\mathscr{S}_D$:

$$\mathscr{S}_D^{\mathrm{inv}} := \big\{F \in \mathscr{S}_D \,:\, \forall_{n\in\mathbb{N}}\, F_n \in \mathcal{O}^{\mathrm{inv}}(\widehat{D}^{\odot n})\big\},$$
$$\mathscr{S}_D^{*} := \big\{F \in \mathscr{S}_D^{\dagger} \,:\, F, F^{\dagger} \in \mathscr{S}_D^{\mathrm{inv}}\big\}.$$

To introduce the *matrix-sum convolution product* $* : \mathscr{S}_D \times \mathscr{S}_D^{\mathrm{inv}} \to \mathscr{S}_D$, reflecting the multiplicative nature of iterated QS integrals, we need some more notations. For $n \in \mathbb{N}$ and $\alpha \subset \{1, \ldots, n\}$ write $\alpha = \{\alpha_1 < \cdots < \alpha_k\}$ and $\{1, \ldots, n\} \setminus \alpha = \{\overline{\alpha}_1 < \cdots < \overline{\alpha}_{n-k}\}$ and define $\Pi_{\alpha;n} \in \mathcal{O}^{*}(\widehat{D}^{\odot n})$ as a certain tensor flip - the linear extension of the map

$$\chi_1 \otimes \cdots \otimes \chi_n \mapsto \chi_{\alpha_1} \otimes \cdots \otimes \chi_{\alpha_k} \otimes \chi_{\overline{\alpha}_1} \otimes \cdots \otimes \chi_{\overline{\alpha}_{n-k}}.$$

If now $F \in \mathscr{S}_D$, define

$$F(\alpha; n) := \Pi^*_{\alpha;n}\big(F_k \otimes I_{n-k}\big)\Pi_{\alpha;n} \in \mathcal{O}(\widehat{D}^{\odot n}), \tag{3.5.2}$$

where to lighten the notation we write I_l for $I_{\widehat{D}^{\odot l}}$ ($l \in \mathbb{N}_0$). Observe that in particular

$$F(\emptyset; n) = F_0 I_n.$$

Additionally let

$$\Delta^{QS}[\alpha; n] := \Pi^*_{\alpha;n}\left((\Delta^{QS})^{\otimes k} \otimes I_{n-k}\right)\Pi_{\alpha;n}.$$

Thus if F_k is a simple tensor $T_1 \otimes \cdots \otimes T_k$, $\alpha = \{\alpha_1 < \ldots < \alpha_k\}$, then $F(\alpha; n) = S_1 \otimes \cdots \otimes S_n$, where

$$S_i = \begin{cases} T_j & \text{if } i \in \alpha, i = \alpha_j \\ I & \text{if } i \notin \alpha \end{cases}.$$

The announced above product $*$ is defined for all pairs $(F, G) \in \mathscr{S}_D \times \mathscr{S}_D^{\mathrm{inv}}$ by

$$(F * G)_n = \sum_{|\boldsymbol{\alpha}| = \{1,\ldots,n\}} F(\alpha_1 \cup \alpha_2; n)\Delta^{QS}[\alpha_2; n]G(\alpha_2 \cup \alpha_3; n) \tag{3.5.3}$$

where $n \in \mathbb{N}_0$ and the sum is over all 3^n disjoint partitions $\alpha_1 \cup \alpha_2 \cup \alpha_3$ of $\{1, \ldots, n\}$. Heuristically, the product $*$ may be equivalently thought of as a sum over all possible fillings of n places with operators coming from F and G with Δ^{QS} intervening whenever chosen operators 'overlap'. To clarify this statement note that for $n = 1$ (3.5.3) gives

$$(F * G)_1 = F_0 G_1 + G_0 F_1 + F_1 \Delta^{QS} G_1.$$

Lemma 3.5.1. *The product enjoys the following properties*:

(a) *if $F, G \in \mathscr{S}_D^{\mathrm{inv}}$ then $F * G \in \mathscr{S}_D^{\mathrm{inv}}$;*

(b) *if* $F \in \mathscr{S}_D$ *and* $G, H \in \mathscr{S}_D^{\mathrm{inv}}$ *then* $F * (G * H) = (F * G) * H$;

(c) *if* $E = 1\delta_0$ *then* $E \in \mathscr{S}_D^*$ *and* $E * F = F * E = F$ *for all* $F \in \mathscr{S}_D$;

(d) *if* $F, G \in \mathscr{S}_D^\dagger$ *with* $G, F^\dagger \in \mathscr{S}_D^{\mathrm{inv}}$ *then* $F*G \in \mathscr{S}_D^\dagger$ *and* $(F*G)^\dagger = G^\dagger * F^\dagger$.

In particular, $(\mathscr{S}_D^*, *)$ *is a unital* **-algebra.*

Proof. To see (b) note that

$$\sum_{|\boldsymbol{\alpha}|=\{1,\dots,n\}} F(\alpha_1 \cup \alpha_2; n)\Delta^{QS}[\alpha_2; n]G(\alpha_2 \cup \alpha_3; n)\Delta^{QS}[\alpha_3; n]H(\alpha_3 \cup \alpha_4; n)$$

is a common expression for $\big(F * (G * H)\big)_n$ and $\big((F * G) * H\big)_n$. The rest is easily verified. ☐

We have already seen in Section 2.4 that for QS purposes a growth condition needs to be imposed on elements of $\mathscr{S}_D$. To obtain algebras of processes we need to restrict further. The choice of restriction here is directly motivated by the Fundamental Theorem on Coalgebras and its consequences expressed in the proof of Lemma 3.3.3. Thus let $\mathcal{H}_D$ denote the set of $F \in \mathscr{S}_D$ satisfying

$$\text{(3.5.4)} \quad \exists_{p,q \in \mathbb{N}, R \subset\subset \mathcal{O}(\widehat{D})} \forall_{n \in \mathbb{N}_0}\ F_n \text{ may be expressed as a sum of } pq^n \text{ terms of the form } X_1 \otimes \cdots \otimes X_n \text{ with } X_1, \dots, X_n \in R,$$

with $\mathcal{H}_D^\dagger, \mathcal{H}_D^{\mathrm{inv}}$ and $\mathcal{H}_D^*$ defined as for $\mathscr{S}$ (the same remark applies to $\mathcal{G}_D^{\mathrm{inv}}$, $\mathcal{G}_D^*$, see (2.4.7), (2.4.8)). It is elementary to check that all of these are subspaces of $\mathcal{G}_D$.

Proposition 3.5.2. *Let* $F \in \mathcal{G}_D$ *and* $G \in \mathcal{H}_D^{\mathrm{inv}}$*. Then* $F * G \in \mathcal{G}_D$*, moreover if* $F \in \mathcal{H}_D$ *then* $F * G \in \mathcal{H}_D$ *too.*

Proof. Let $H = F * G$ and choose p, q and R for G according to (3.5.4). Let $S \subset\subset \widehat{D}$ and let $n \in \mathbb{N}$ and $\chi_1, \dots, \chi_n \in S$. Then, for any partition $\alpha \cup \beta \cup \gamma$ of $\{1, \dots, n\}$,

$$F(\alpha \cup \beta; n)\Delta^{QS}[\beta; n]G(\beta \cup \gamma; n)(\chi_1 \otimes \cdots \otimes \chi_n)$$

is a sum of $pq^{\#(\beta\cup\gamma)}$ terms of the form

$$F(\alpha\cup\beta;n)(\eta_1\otimes\cdots\otimes\eta_n)$$

where each η_i belongs to the finite set $S' := RS\cup\Delta^{QS}RS\cup S$. Thus, choosing C_1 and $C_2\geq 1$ for the pair (F,S') according to (2.4.7), and setting $M = \max\{\|\eta\| : \eta\in S'\}$,

$$\|H_n(\chi_1\otimes\cdots\otimes\chi_n)\| \leq \sum pq^{\#(\beta\cup\gamma)}C_1C_2^{\#(\alpha\cup\beta)}M^{\#\gamma} = C_1'(C_2')^n,$$

where $C_1' = pC_1$ and $C_2' = 3(C_2+qC_2+qM)$. Thus $H\in\mathcal{G}_D$.

If $F\in\mathcal{H}_D$ then, choosing p',q' and R' for F (and assuming without loss that $I\in R\cap R'$), $F(\alpha\cup\beta;n)\Delta^{QS}[\beta;n]G(\beta\cup\gamma;n)$ is a sum of $p'(q')^{\#(\alpha\cup\beta)}pq^{\#(\beta\cup\gamma)}$ terms of the form $Z_1\otimes\cdots\otimes Z_n$ where $Z_i\in R'R\cup R'\Delta^{QS}R$. Thus H_n is a sum of $pp'3^n(q+qq'+q')^n$ terms of this form and $H\in\mathcal{H}_D$. $\square$

As an immediate consequence we have the following result.

Theorem 3.5.3. *$(\mathcal{H}_D^*,*)$ is a unital $*$-subalgebra of $(\mathscr{S}_D^*,*)$.*

Multiplicativity for iterated QS integrals

In this subsection we show how the product $*$ introduced above corresponds to the multiplicative structure of iterated QS integrals.

Elementary inductive argument using (2.4.4) yields the following fact (for a more sophisticated version, relevant for C^*-algebraic processes see Theorem 2.2 of [LW$_4$]; it is an essential tool leading to Theorems 4.6.2 and 4.6.3).

Fact 3.5.4. *Let $F\in\mathcal{G}_D^\dagger$ and $G\in\mathcal{G}_D^{\mathrm{inv}}$. Then, for each $n\in\mathbb{N}$, $f,g\in\mathbb{S}_D$,*

$$\sum_{i,j=0}^{n}\left\langle\Lambda_t^i(F_i)\varepsilon(f),\Lambda_t^j(G_j)\varepsilon(g)\right\rangle = \sum_{k=0}^{2n}\left\langle\varepsilon(f),\Lambda_t^k(H_k)\varepsilon(g)\right\rangle$$

where $H = F^\dagger * G$.

Here is the relevant consequence of the above fact for the integral Λ encompassing all elements of the sequence at once. Recall the partially defined product (3.5.1).

Proposition 3.5.5. *Let $F \in \mathcal{G}_D^\dagger$ and $G \in \mathcal{G}_D^{\mathrm{inv}}$ be such that $F * G \in \mathcal{G}_D$. Then, for each $t \geq 0$, $\mathrm{Dom}\,\Lambda_t(F)^{\dagger *} \supset \mathrm{Ran}\,\Lambda_t(G)$ and*

$$\Lambda_t(F * G) = \Lambda_t(F) \cdot \Lambda_t(G).$$

Proof. For $H \in \mathscr{S}_D$ and $n \in \mathbb{N}_0$ write $\Lambda^{[n]}(H)$ for $\sum_{i=0}^n \Lambda^i(H_i) \in \mathbb{P}_{\mathrm{c}}(\mathcal{E}_D)$, so that if $H \in \mathcal{G}_D$ then for each $f \in \mathbb{S}_D$

$$\Lambda_t^{[n]}(H)\varepsilon(f) \overset{n\to\infty}{\longrightarrow} \Lambda_t(H)\varepsilon(f).$$

By the fact above and (2.4.11), for all $f, g \in \mathbb{S}_D$

$$\begin{aligned}\langle \Lambda_t(F)^\dagger \varepsilon(f), \Lambda_t(G)\varepsilon(g)\rangle &= \lim_{n\to\infty} \left\langle \varepsilon(f), \Lambda_t^{[n]}(F * G)\varepsilon(g)\right\rangle \\ &= \langle \varepsilon(f), \Lambda_t(F * G)\varepsilon(g)\rangle,\end{aligned}$$

and so the result follows. $\square$

The following theorem summarises the content of this subsection and the previous one; it is an immediate consequence of Proposition 3.5.5 and Theorem 3.5.3.

Theorem 3.5.6. *Let $\mathcal{P} = \Lambda(\mathcal{H}_D^*) \subset \mathbb{P}_{\mathrm{Hc}}^\dagger(\mathcal{E}_D)$. Then, with respect to the product defined in (3.5.1) — extended pointwise, the map Λ restricts to a unital *-algebra isomorphism of unital *-algebras:*

$$(\mathcal{H}_D^*, *) \to (\mathcal{P}, \cdot).$$

Multiplicative QS convolution cocycles

As mentioned before, we fix a bialgebra $\mathcal{A}$.

Definition 3.5.7. A process $k \in \mathbb{P}^\dagger(\mathcal{A}; \mathcal{E}_D)$ is called weakly multiplicative if for all $a, b \in \mathcal{A}, t \geq 0$

$$k_t(ab) = k_t(a) \cdot k_t(b).$$

It is easy to see that by the arguments above the question of multiplicativity for a stochastically generated QS convolution cocycle $l^\varphi \in \mathbb{QSCC}(\mathcal{A}; \mathcal{E}_D)$ is equivalent to that of the multiplicativity of the map $\widetilde{\upsilon}^\varphi : \mathcal{A} \to \mathcal{H}_D^*$ derived from $\varphi \in L(\mathcal{A}; \mathcal{O}(D))$ (as described in Section 3.3). The following proposition shows that the latter question may be reduced to a simple statement concerning properties of φ.

Proposition 3.5.8. *Let $\upsilon = \upsilon^\varphi$, $\widetilde{\upsilon} = \widetilde{\upsilon}^\varphi$ for $\varphi \in L\big(\mathcal{A}; \mathcal{O}^{\mathrm{inv}}(\widehat{D})\big)$. Then the following are equivalent:*

(i) $\varphi(ab) = \varphi(a)\epsilon(b) + \epsilon(a)\varphi(b) + \varphi(a)\Delta^{QS}\varphi(b)$ *for all* $a, b \in \mathcal{A}$.

(ii) $\upsilon(ab) = \upsilon(a) * \upsilon(b)$ *for all* $a, b \in \mathcal{A}$.

(iii) $\widetilde{\upsilon}(ab) = \widetilde{\upsilon}(a) * \widetilde{\upsilon}(b)$ *for all* $a, b \in \mathcal{A}$.

Proof. (i) is contained in (ii) since $\upsilon_1(ab) = \varphi(ab)$ and $\big(\upsilon(a) * \upsilon(b)\big)_1$ is the right hand side of (i). Thus (ii) implies (i). Conversely, if (i) holds then $\mathcal{P}(1)$ holds where $\mathcal{P}(n)$ is the proposition

$$\forall_{a,b\in\mathcal{A}} \quad \upsilon_n(ab) = \big(\upsilon(a) * \upsilon(b)\big)_n.$$

By the multiplicativity of $\epsilon = \upsilon_0$, $\mathcal{P}(0)$ holds. Assume therefore that $\mathcal{P}(k)$ holds for $k \leq n$, and fix $a, b \in \mathcal{A}$. Employing Sweedler notation and using (3.3.6), $\mathcal{P}(1)$ and then $\mathcal{P}(n)$,

$$\begin{aligned}
&\upsilon_{n+1}(ab) \\
&= \varphi(a_{(1)}b_{(1)}) \otimes \upsilon_n(a_{(2)}b_{(2)}) \\
&= \big[\varphi(a_{(1)})\epsilon(b_{(1)}) + \epsilon(a_{(1)})\varphi(b_{(1)}) + \varphi(a_{(1)})\Delta^{QS}\varphi(b_{(1)})\big] \otimes \\
&\qquad\qquad \sum_{|\boldsymbol{\alpha}|=\{1,\ldots,n\}} \upsilon(a_{(2)})(\alpha_1 \cup \alpha_2; n)\Delta^{QS}[\alpha_2; n]\upsilon(b_{(2)})(\alpha_2 \cup \alpha_3; n),
\end{aligned}$$

where the sum is over all partitions of the set $\{1,\dots,n\}$ into three disjoint subsets. The identity

$$\varphi(c_{(1)}) \otimes \upsilon(c_{(2)})(\lambda; n) = \upsilon(c)(\overset{\bullet\!\to}{\lambda}; n+1),$$

(for $c \in \mathcal{A}$, $\lambda \subset \{1,\dots,n\}$) gives the following equalities

$$\begin{aligned}\varphi(c_{(1)}) \otimes \upsilon(c_{(2)})(\lambda; n)\Delta^{QS}[\lambda \cap \mu; n]\upsilon(d)(\mu; n)&\\ = \upsilon(c)(\overset{\bullet\!\to}{\lambda}; n+1)\Delta^{QS}[\overset{\circ\!\to}{\nu}; n+1]\upsilon(d)(\overset{\circ\!\to}{\mu}; n+1)&,\end{aligned}$$

and

$$\begin{aligned}\varphi(c_{(1)})\Delta^{QS}\varphi(d_{(1)}) \otimes \upsilon(c_{(2)})(\lambda; n)\Delta^{QS}[\lambda \cap \mu; n]\upsilon(d_{(2)})(\mu; n)&\\ = \upsilon(c)(\overset{\bullet\!\to}{\lambda}; n+1)\Delta^{QS}[\overset{\bullet\!\to}{\nu}; n+1]\upsilon(d)(\overset{\bullet\!\to}{\mu}; n+1)&,\end{aligned}$$

in which $\nu = \lambda \cap \mu$. Thus

$$\begin{aligned}&\upsilon_{n+1}(ab)\\ &= \sum_{|\boldsymbol{\alpha}|=\{1,\dots,n\}} \Big(\upsilon(a)(\overset{\bullet\!\to}{\alpha_1} \cup \overset{\circ\!\to}{\alpha_2}; n+1)\Delta^{QS}[\overset{\circ\!\to}{\alpha_2}; n+1]\upsilon(b)(\overset{\circ\!\to}{\alpha_2} \cup \overset{\circ\!\to}{\alpha_3}; n+1)\\ &\qquad + \upsilon(a)(\overset{\circ\!\to}{\alpha_1} \cup \overset{\circ\!\to}{\alpha_2}; n+1)\Delta^{QS}[\overset{\circ\!\to}{\alpha_2}; n+1]\upsilon(b)(\overset{\circ\!\to}{\alpha_2} \cup \overset{\bullet\!\to}{\alpha_3}; n+1)\\ &\qquad + \upsilon(a)(\overset{\circ\!\to}{\alpha_1} \cup \overset{\bullet\!\to}{\alpha_2}; n+1)\Delta^{QS}[\overset{\bullet\!\to}{\alpha_2}; n+1]\upsilon(b)(\overset{\bullet\!\to}{\alpha_2} \cup \overset{\circ\!\to}{\alpha_3}; n+1)\Big)\\ &= \big(\upsilon(a) * \upsilon(b)\big)_{n+1}.\end{aligned}$$

The implication (i)$\Rightarrow$(ii) therefore follows by induction.

For (ii)$\Leftrightarrow$(iii) note the following general fact. Let $F, G \in \mathscr{S}^{\mathrm{inv}}_{\widehat{D}}$, and assume $\widetilde{F}, \widetilde{G} \in \mathscr{S}^{\mathrm{inv}}_{\widehat{D}}$ are constructed from F, G by composing with the tensor flips τ_n reversing the order of the copies of $\widehat{D}$ in $\mathcal{O}(\widehat{D}^{\odot n})$ for each $n \in \mathbb{N}$. For any

$k \in \mathbb{N}$

$$\tau_k \circ \ (F * G)_k = \tau_k \circ \left(\sum_{|\boldsymbol{\alpha}|=\{1,\ldots,k\}} F(\alpha_1 \cup \alpha_2; k)\Delta^{QS}[\alpha_2; k]G(\alpha_2 \cup \alpha_3; k) \right)$$

$$= \sum_{|\boldsymbol{\alpha}|=\{1,\ldots,k\}} \widetilde{F}(\widetilde{\alpha}_1 \cup \widetilde{\alpha}_2; k)\Delta^{QS}[\widetilde{\alpha}_2; k]\widetilde{G}(\widetilde{\alpha}_2 \cup \widetilde{\alpha}_3; k) = (\widetilde{F} * \widetilde{G})_k,$$

where $\widetilde{\alpha}_1$ denotes the set $\{k-i : i \in \alpha_1\}$ and $\widetilde{\alpha}_2, \widetilde{\alpha}_3$ are defined in an analogous way. The formula above applied to $\upsilon(a)$ and $\upsilon(b)$ (or respectively $\widetilde{\upsilon}(a)$ and $\widetilde{\upsilon}(b)$) in place of F and G ends the proof. □

Proposition 3.5.9. *Let $\varphi \in L\big(\mathcal{A}; \mathcal{O}^*(\widehat{D})\big)$ and set $k = l^\varphi$.*

(a) *If k is weakly multiplicative then, for all $a, b \in \mathcal{A}$,* $\operatorname{Dom}\big(\varphi(a)^\dagger\big)^* \supset \operatorname{Ran}\Delta^{QS}\varphi(b)$ *and*

$$\varphi(ab) = \varphi(a)\epsilon(b) + \epsilon(a)\varphi(b) + \varphi(a) \cdot \Delta^{QS}\varphi(b). \tag{3.5.5}$$

(b) *Conversely, if φ satisfies* (3.5.5) *then k is weakly multiplicative.*

Proof. Let $\widetilde{\upsilon} = \widetilde{\upsilon}^\varphi$. If k is weakly multiplicative then, for any $a, b \in \mathcal{A}$, using (3.3.10) and the Second Fundamental Formula,

$$\left\langle \hat{c}, \varphi(ab)\hat{d} \right\rangle = \left\langle \varphi(a)^\dagger\hat{c}, \epsilon(b)\hat{d} \right\rangle + \left\langle \overline{\epsilon(a)}\hat{c}, \varphi(b)\hat{d} \right\rangle + \left\langle \varphi(a)^\dagger\hat{c}, \Delta^{QS}\varphi(b)\hat{d} \right\rangle$$

for all $c, d \in D$. By sesquilinearity, this implies that

$$\langle \varphi(a)^\dagger\chi, \Delta^{QS}\varphi(b)\eta \rangle = \big\langle \chi, \big[\varphi(ab) - \varphi(a)\epsilon(b) - \epsilon(a)\varphi(b)\big]\eta \big\rangle$$

for all $\chi, \eta \in \widehat{D}$, and so (a) holds.

Conversely, if φ satisfies (3.5.5) then, by Lemma 3.3.1, $\widetilde{\upsilon}$ is $\mathcal{H}^*_D$-valued and, by Proposition 3.5.8, $\widetilde{\upsilon}(a) * \widetilde{\upsilon}(b) = \widetilde{\upsilon}(ab)$. Thus, by Theorem 3.5.6, k satisfies $\operatorname{Dom}(k_t(a)^\dagger)^* \supset \operatorname{Ran} k_t(b)$ and

$$k_t(a) \cdot k_t(b) = \Lambda_t\big(\widetilde{\upsilon}(a) * \widetilde{\upsilon}(b)\big) = \Lambda_t\big(\widetilde{\upsilon}(ab)\big) = k_t(ab),$$

so k is weakly multiplicative. □

In view of Theorem 3.5.6, the following characterisation is obtained from Lemma 3.3.5 and Proposition 3.5.9. Its origins date back to the paper [Glo] and the book [Sch]. Recall the algebra of processes defined in Theorem 3.5.6, and the remark on block matrix forms after Lemma 3.3.5.

Theorem 3.5.10. *Let $k = l^{\varphi}$, where $\varphi \in L(\mathcal{A}; \mathcal{O}^*(\widehat{D}))$, and suppose that $\mathcal{A}$ is a unital *-bialgebra. Then the following are equivalent*:

(i) *k is unital and *-homomorphic as a map $\mathcal{A} \to (\mathcal{P}, \boldsymbol{\cdot})$;*

(ii) *φ vanishes at $1_{\mathcal{A}}$ and satisfies*

$$\varphi(a^*b) = \varphi(a)^*\epsilon(b) + \overline{\epsilon(a)}\varphi(b) + \varphi(a)^*\Delta^{QS}\varphi(b); \tag{3.5.6}$$

(iii) *φ has block matrix form*

$$\begin{bmatrix} \gamma & \delta^{\dagger} \\ \delta & \rho - \iota \circ \epsilon \end{bmatrix} \tag{3.5.7}$$

in which ι is the ampliation $z \mapsto zI_D$;

$$\rho : \mathcal{A} \to \mathcal{O}^*(D) \text{ is a unital *-homomorphism}; \tag{3.5.8}$$

$\delta : \mathcal{A} \to |D\rangle$ is a (ρ, ϵ)-derivation:

$$\delta(ab) = \delta(a)\epsilon(b) + \rho(a)\delta(b); \tag{3.5.9}$$

$\gamma : \mathcal{A} \to \mathbb{C}$ is linear and satisfies

$$\gamma(a^*b) = \overline{\gamma(a)}\epsilon(b) + \overline{\epsilon(a)}\gamma(b) + \delta(a)^*\delta(b). \tag{3.5.10}$$

Following P.-A. Meyer ([Mey]) we shall refer to such (γ, δ, ρ) as a *D-Schürmann triple* on $\mathcal{A}$.

3.6 Quantum Lévy processes

In this section Schürmann's theorem on the reconstruction of a quantum Lévy process from its 'generator' is described. A new simple proof of the existence of an equivalent realisation of each quantum Lévy process on a Fock space is given, based on the results established in previous sections.

Let $\mathcal{A}$ be a unital *-bialgebra.

Definition 3.6.1 ([ASW], [Sch]). By a *quantum Lévy process* on $\mathcal{A}$ over a unital *-algebra-with-state $(\mathcal{B}, \omega)$ is meant a family $\{j_{s,t} : \mathcal{A} \to \mathcal{B} \, : \, 0 \leq s \leq t\}$ of unital *-homomorphisms satisfying

(QL1) $j_{r,t} = j_{r,s} \star j_{s,t}$ for $0 \leq r \leq s \leq t$;

(QL2) $j_{t,t}(a) = \epsilon(a) 1_{\mathcal{B}}$ for $t \geq 0$, $a \in \mathcal{A}$;

(QL3) $\{j_{s_i,t_i}(\mathcal{A}) : i = 1, \ldots n\}$ commute and

$$\omega\left(\prod_{i=1}^{n} j_{s_i,t_i}(a_i)\right) = \prod_{i=1}^{n} \omega\big(j_{s_i,t_i}(a_i)\big),$$

whenever $n \in \mathbb{N}$, the intervals $[s_i, t_i), \ldots, [s_n, t_n)$ are disjoint and $a_1, \ldots, a_n \in \mathcal{A}$;

(QL4) $\omega \circ j_{s,t} = \omega \circ j_{0,t-s}$ for $0 \leq s \leq t$;

(QL5) $\omega \circ j_{0,t}(a) \xrightarrow{t \to 0^+} \epsilon(a)$ for all $a \in \mathcal{A}$.

Condition (QL1) is known as the *increment property*; the others respectively as the initial condition, (*tensor*) *independence of increments*, *time-homogeneity* and *continuity*. It is immediately verified that

$$\kappa_t := \omega \circ j_{0,t}$$

defines a continuous convolution semigroup of states on $\mathcal{A}$, called the *one-dimensional distribution* of the quantum Lévy process; its generator is also

referred to as the *generator* of the quantum Lévy process. For more information on quantum Lévy processes on *-bialgebras, generalisations to free, boolean and monotone case, connections with Lie algebras and many examples we refer to the book [Sch] and the lecture notes [Fra].

Quantum Lévy processes j^i on $\mathcal{A}$ over $(\mathcal{B}^i, \omega^i)$ $(i = 1, 2)$ are said to be *equivalent* if they satisfy

$$\omega^1 \left(\prod_{k=1}^{n} j^1_{s_k,t_k}(a_k) \right) = \omega^2 \left(\prod_{k=1}^{n} j^2_{s_k,t_k}(a_k) \right)$$

for all $n \in \mathbb{N}$, disjoint intervals $[s_k, t_k)$ and elements a_k $(k = 1, \ldots, n)$. In view of the axioms (QL1-4) it is clear that two quantum Lévy processes are equivalent if and only if their one-dimensional distributions coincide — equivalently, if their generators are equal.

Let $\varphi \in L\left(\mathcal{A}; \mathcal{O}^*(\widehat{D})\right)$, and $k = l^\varphi \in \mathbb{QSCC}^\dagger_{\mathrm{Hc}}(\mathcal{A}; \mathcal{E}_D)$ be unital, real and weakly multiplicative. Then, setting

$$\begin{aligned} \mathcal{A}^\varphi = \mathrm{Lin}\Big\{ & k_{s_1}(a_1) \cdot \sigma_{s_1}\big(k_{s_2 - s_1}(a_2)\big) \cdot \cdots \cdot \sigma_{s_{n-1}}\big(k_{s_n - s_{n-1}}(a_n)\big) : \\ & \qquad n \in \mathbb{N}, 0 \le s_1 \le \cdots \le s_n, a_1, \ldots, a_n \in \mathcal{A} \Big\}, \\ j^\varphi_{s,t} &= \sigma_s \circ k_{t-s} : \mathcal{A} \to \mathcal{A}^\varphi, \text{ and} \\ \omega^\varphi &= \omega_\Omega|_{\mathcal{A}^\varphi}, \end{aligned}$$

$\mathcal{A}^\varphi$ is a unital *-algebra in the involutive linear space $\mathcal{O}^\dagger(\mathcal{E}_D)$ with product given by (3.5.1), ω^φ is a state on $\mathcal{A}^\varphi$ and it is easily checked that j^φ is a quantum Lévy process over $(\mathcal{A}^\varphi, \omega^\varphi)$ with generator γ, where $\gamma = \varphi^0_0$ (the top-left component of the block matrix form of φ). Let us call this type a *Fock space quantum Lévy process.*

Note that since a quantum Lévy process is unital (real) and positive, its generator γ vanishes on $1_\mathcal{A}$, is real and conditionally positive (see (3.1.10)).

Theorem 3.6.2 ([Sch]). *Let γ be a real, conditionally positive linear functional on $\mathcal{A}$ vanishing at $1_\mathcal{A}$. Then there is a Fock space quantum Lévy process*

with generator γ.

Proof. The proof follows by a GNS-type construction applied to γ viewed as a positive functional on $\operatorname{Ker}\epsilon$. Set $D = \operatorname{Ker}\epsilon \big/ N$ where

$$N = \big\{ a \in \operatorname{Ker}\epsilon \,:\, \gamma(a^*a) = 0 \big\}.$$

Then $\big([a],[b]\big) \mapsto \gamma(a^*b)$ defines an inner product on D; let k be its completion. Then $\rho(a) : [c] \mapsto [ac]$ defines for each $a \in \mathcal{A}$ an operator on D. It is obvious that ρ is a unital representation of $\mathcal{A}$ on D satisfying

$$\langle \rho(a)[b],[c] \rangle = \langle [b], \rho(a^*)[c] \rangle .$$

Thus ρ is a unital *-homomorphism $\mathcal{A} \to \mathcal{O}^*(D)$. Moreover the linear map $\delta : a \mapsto |d(a)\rangle$, where $d(a) = [a - \epsilon(a)I]$, is easily seen to be a (ρ,ϵ)-derivation $\mathcal{A} \to |D\rangle$ satisfying

$$\delta(a)^*\delta(b) = \gamma(a^*b) - \overline{\gamma(a)}\epsilon(b) - \overline{\epsilon(a)}\gamma(b).$$

Set $k = l^\varphi$, where φ is the map $\mathcal{A} \to \mathcal{O}^*(\widehat{D})$ with block matrix form given by the prescription (3.5.7). Then Theorem 3.5.10 implies that k is *-homomorphic (i.e. real and weakly multiplicative) and unital. Since $\varphi_0^0 = \gamma$ the result follows. $\square$

Corollary 3.6.3. *Every quantum Lévy process is equivalent to a Fock space quantum Lévy process.*

3.7 Perturbation

The section is concerned with the perturbation of *-homomorphic QS convolution cocycles by unitary Weyl cocycles.

Consider first the case of the trivial bialgebra $\mathbb{C}$, and let $\varphi \in L(\mathbb{C}; \mathcal{O}(\widehat{D}))$. Then φ and l^φ are determined by the operator $L := \varphi(1) \in \mathcal{O}(\widehat{D})$ and the

operator Markovian cocycle $X^L := l^\varphi(1) \in \mathbb{P}_{\mathrm{Hc}}(\mathcal{E}_D)$ which satisfies the operator QS differential equation

$$\mathrm{d}X_t = (L \otimes X_t)\,\mathrm{d}\Lambda_t, \quad X_0 = I. \tag{3.7.1}$$

These processes have explicit action on exponential vectors: for any $f \in \mathbb{S}_D$

$$X_t^L \varepsilon(f) = \exp\Big(tz + \int_0^t \beta(f(s))\,ds\Big)\varepsilon\Big((Rf)_{[0,t[} + d_{[0,t[}\Big) \tag{3.7.2}$$

where

$$\begin{bmatrix} z & \beta \\ |d\rangle & R - I \end{bmatrix}, \text{ with } z \in \mathbb{C},\ d \in \mathsf{k},\ \beta \in L(D;\mathbb{C}),\ \text{and } R \in \mathcal{O}(D),$$

is the block matrix form of L. From either of the above descriptions it is clear that the map $L \mapsto X^L$ is injective $\mathcal{O}(\widehat{D}) \to \mathbb{P}_{\mathrm{Hc}}(\mathcal{E}_D)$. Moreover if $L \in \mathcal{O}^\dagger(\widehat{D})$ (equivalently, $R \in \mathcal{O}^\dagger(D)$ and $\beta = \langle c|$ for some $c \in \mathsf{k}$) then $X^L \in \mathbb{P}^\dagger_{\mathrm{Hc}}(\mathcal{E}_D)$ and $(X^L)^\dagger = X^{L^\dagger}$. Similarly, if $L \in \mathcal{O}^{\mathrm{inv}}(\widehat{D})$ (equivalently, $R \in \mathcal{O}^{\mathrm{inv}}(D)$ and $d \in D$) then $X_t^L \in \mathcal{O}^{\mathrm{inv}}(\mathcal{E}_D)$ for each t. If $L \in \mathcal{O}(\widehat{D})$ and $M \in \mathcal{O}^{\mathrm{inv}}(\widehat{D})$ then, by the explicit action (3.7.2),

$$X^L X^M = X^{L \blacklozenge M} \tag{3.7.3}$$

where

$$L \blacklozenge M := L + M + L\Delta^{QS} M \tag{3.7.4}$$

By the above injectivity $\big(\mathcal{O}^{\mathrm{inv}}(\widehat{D}), \blacklozenge\big)$ is a semigroup with identity 0; clearly $\big(\mathcal{O}^*(\widehat{D}), \blacklozenge\big)$ is an involutive semigroup: $(L \blacklozenge M)^\dagger = M^\dagger \blacklozenge L^\dagger$. Note that these identities contain the Weyl commutation relations.

The above formula implies that for $L \in \mathcal{O}^*(\widehat{D})$

$$X^L \text{ is isometric} \iff L^\dagger \blacklozenge L = 0, \text{ whereas } X^L \text{ is coisometric} \iff L \blacklozenge L^\dagger = 0,$$

cf. analogous characterisations described in [LW$_2$].

In the next proposition, (3.7.4) is extended by left and right actions of (parts of) $\mathcal{O}^*(\widehat{D})$ on $L\big(\mathcal{C};\mathcal{O}(\widehat{D})\big)$, for a coalgebra $\mathcal{C}$.

Proposition 3.7.1. *Let* $\varphi \in L(\mathcal{C};\mathcal{O}(\widehat{D}))$ *and let* $L, M \in \mathcal{O}(\widehat{D})$*.*

(a) *If* $\varphi \in L\left(\mathcal{C};\mathcal{O}^\dagger(\widehat{D})\right)$ *and* $M \in \mathcal{O}^{\text{inv}}(\widehat{D})$ *then*

$$l^\varphi(\,\cdot\,)X^M = l^{\varphi\blacklozenge M},$$

(b) *If* $L \in \mathcal{O}^\dagger(\widehat{D})$ *and* $\varphi \in L\left(\mathcal{C};\mathcal{O}^{\text{inv}}(\widehat{D})\right)$ *then*

$$l_t^\varphi(\mathcal{C}) \subset \text{Dom}(X_t^L)^{\dagger *} \text{ and } X^L \boldsymbol{\cdot} l^\varphi(\,\cdot\,) = l^{L\blacklozenge\varphi},$$

where for each $a \in \mathcal{C}$

$$(\varphi\blacklozenge M)(a) := \varphi(a)(I + \Delta^{QS}M) + \epsilon(a)M$$

and

$$(L\blacklozenge\varphi)(a) := (I + L\Delta^{QS})\varphi(a) + \epsilon(a)L.$$

Proof. These follow easily from the two Fundamental Formulae (2.4.1) and (2.4.4). ∎

The above formulae extend (3.7.4) – this may be checked by setting $\mathcal{C} = \mathbb{C}$ and $a = 1$.

Let $\varphi \in L\big(\mathcal{C};\mathcal{O}^*(\widehat{D})\big)$ and $L_1, L_2 \in \mathcal{O}^*(\widehat{D})$. Then their block matrix forms (see (3.3.11)) are respectively given by

$$\begin{bmatrix} \gamma & \alpha \\ \chi & \nu - \iota\circ\epsilon \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z_i & \langle c_i| \\ |d_i\rangle & R_i - I \end{bmatrix},$$

where $z_i \in \mathbb{C}$, $c_i \in D$, $R_i \in \mathcal{O}^*(D)$, and

$$\begin{aligned}
&(L_1^\dagger \blacklozenge \varphi \blacklozenge L_2)(a) \\
&= (I + \Delta^{QS} L_1)^\dagger \varphi(a)(I + \Delta^{QS} L_2) + \epsilon(a) L_1^\dagger \blacklozenge L_2 \\
&= \begin{bmatrix} \widetilde{\gamma}(a) & \big(\alpha(a) + \langle d_1|\nu(a)\big) R_2 + \epsilon(a)\langle c_2| \\ R_1^\dagger\big(\chi(a) + \nu(a)|d_2\rangle\big) + \epsilon(a)|c_1\rangle & R_1^\dagger \nu(a) R_2 - \epsilon(a) I \end{bmatrix}
\end{aligned}$$

where

$$\widetilde{\gamma}(a) = \gamma(a) + (z_1^* + z_2)\epsilon(a) + \alpha(a)|d_2\rangle + \langle d_1|\chi(a) + \big\langle d_1, \nu(a) d_2\big\rangle.$$

Let now $\mathcal{A}$ be a unital *-bialgebra and consider conjugation by a single map $L \in \mathcal{O}^*(\widehat{D})$:

$$\widetilde{\varphi} = L^\dagger \blacklozenge \varphi \blacklozenge L,$$

where $\varphi \in L\big(\mathcal{A}; \mathcal{O}^*(\widehat{D})\big)$. It is easily checked that if l^φ is real then $l^{\widetilde{\varphi}}$ is real; if l^φ is unital then

$$l^{\widetilde{\varphi}} \text{ is unital } \iff X^L \text{ is isometric;}$$

and if l^φ is weakly multiplicative then $l^{\widetilde{\varphi}}$ is weakly multiplicative if and only if for all $a \in \mathcal{A}$

$$(\Delta^{QS} L + I)^* \big(\Delta^{QS}\varphi(a) + \epsilon(a) I\big)^\dagger L \blacklozenge L^\dagger \big(\Delta^{QS}\varphi(a) + \epsilon(a) I\big)(\Delta^{QS} L + I) = 0.$$

Therefore, considering perturbations by unitary (Weyl) cocycles, one obtains the action of the Euclidean group of D on Schürmann triples associated with unital *-homomorphic QS convolution cocycles on $\mathcal{A}$ (cf. [Fra]). This action has a simple matricial description: if

$$L = \begin{bmatrix} i\mu - \frac{1}{2}\|v\|^2 & -\langle v| V \\ |v\rangle & V - I \end{bmatrix},$$

where $\mu \in \mathbb{R}, v \in D$ and $V \in \mathcal{O}^*(D)$ is unitary, then for all $a \in \mathcal{A}$

$$\widetilde{\varphi}(a) = \begin{bmatrix} 1 & \langle v| \\ 0 & V^* \end{bmatrix} \varphi(a) \begin{bmatrix} 1 & 0 \\ |v\rangle & V \end{bmatrix}.$$

Thus if

$$\varphi = \begin{bmatrix} \gamma & \delta^\dagger \\ \delta & \rho - \iota \circ \epsilon \end{bmatrix}$$

then for all $a \in \mathcal{A}$

$$\begin{aligned}
&\widetilde{\gamma}(a) = \gamma(a) + \delta^\dagger(a)|v\rangle + \langle v|\delta(a) + \big\langle v, \big(\rho(a) - \epsilon(a)I\big)v\big\rangle, \\
&\widetilde{\delta}(a) = V^*\Big(\delta(a) + \big(\rho(a) - \epsilon(a)I\big)|v\rangle\Big) \text{ and} \\
&\widetilde{\rho}(a) = V^*\rho(a)V.
\end{aligned}$$

Notice that the part of the action determined by V is trivial in the sense that only a unitary transformation of the Schürmann triple (λ, δ, ρ) leaving λ invariant is effected, so that the perturbed quantum Lévy process $l^{\widetilde{\varphi}}$ is equivalent to the unperturbed one l^{φ}. For nonzero v the perturbation still does not change the characteristics of the quantum Lévy process, in the sense that *Gaussian* processes remain Gaussian and the same is true for *Poisson* and *drift* processes (for relevant definitions see [Fra]).

3.8 Opposite QS convolution cocycles

This section is concerned with the opposite counterpart of the notion of a QS convolution cocycle introduced in Section 3.2.

Definition 3.8.1. An opposite quantum stochastic convolution cocycle (on $\mathcal{C}$ with domain $\mathcal{E}_D$) is a process $l \in \mathbb{P}(\mathcal{C}; \mathcal{E}_D)$ such that, for all $s, t \geq 0$,

$$l_{s+t} = (\sigma_s \circ l_t) \star l_s \tag{3.8.1}$$

and for all $a \in \mathcal{C}$

$$l_0(a) = \epsilon(a) I_{\mathcal{E}_D}.$$

Family of all opposite QS convolution cocycles is denoted by $\mathbb{QSCC}^{\mathrm{opp}}(\mathcal{C};\mathcal{E}_D)$; the notation adorned with subscripts and superscript † according to our convention.

Remark 3.8.2. The opposite convolution increment formula makes use of the identification $\mathcal{F}_{[0,s+t)} \cong \mathcal{F}_{[s,s+t)} \otimes \mathcal{F}_{[0,s)}$. The weak description of the opposite convolution increment property takes the form:

$$\mathrm{e}^{-\langle f_{[0,t)}, g_{[0,t)}\rangle} \left\langle \varepsilon(f_{[0,t)}), l_t(a)\varepsilon(g_{[0,t)})\right\rangle = \prod_{i=0}^{n-1} \lambda_{t_{i+1}-t_i}^{f(t_i),g(t_i)}(a_{(n-i)}) \tag{3.8.2}$$

(compare with the formula (3.2.3) of Lemma 3.2.3).

Opposite cocycles also have associated convolution semigroups, given by formula (3.2.4); two opposite QS convolution cocycles with identical convolution semigroups are equal.

Opposite QS convolution cocycles arise as solutions of the opposite coalgebraic QS differential equations of the form:

$$dk_t = \varphi \star k_t \, d\Lambda_t, \quad k_0 = \iota \circ \epsilon, \tag{3.8.3}$$

where $\varphi \in L\big(\mathcal{C}; \mathcal{O}(\widehat{D})\big)$.

Below we formulate the relevant opposite versions of basic theorems proved earlier for QS convolution cocycles.

Theorem 3.8.3. *Let $\varphi \in L\left(\mathcal{C}; \mathcal{O}(\widehat{D})\right)$ and set $\upsilon = \upsilon^\varphi$. Then the process ${}^\varphi l := \Lambda \circ \upsilon$ strongly satisfies the opposite coalgebra quantum stochastic differential equation* (3.8.3). *It is a unique weak solution of* (3.8.3).

Theorem 3.8.4. *Let $k \in \mathbb{P}(\mathcal{C};\mathcal{E}_D)$. Then the following are equivalent :*

(i) $k \in \mathbb{QSCC}_{\mathrm{Hc}}^{\dagger\ \mathrm{opp}}(\mathcal{C};\mathcal{E}_D)$;

(ii) *$k = {}^{\varphi}l$ for some $\varphi \in L\big(\mathcal{C}; \mathcal{O}^{\dagger}(\widehat{D})\big)$.*

Theorem 3.8.5. *Let $k = {}^{\varphi}l$, where $\varphi \in L(\mathcal{A}; \mathcal{O}^*(\widehat{D}))$, and suppose that $\mathcal{A}$ is a unital *-bialgebra. Then the following are equivalent:*

(i) *k is unital and *-homomorphic as a map $\mathcal{A} \to (\mathcal{P}, \cdot\,)$.*

(ii) *φ vanishes at $1_{\mathcal{A}}$ and satisfies*

$$\varphi(a^*b) = \varphi(a)^*\epsilon(b) + \overline{\epsilon(a)}\varphi(b) + \varphi(a)^*\Delta^{QS}\varphi(b).$$

The proofs of the above may be conducted exactly along the same lines as for usual QS convolution cocycles - whenever it was convenient, the arguments in the proofs were formulated to make this transparent (for example by working with both υ and $\widetilde{\upsilon}$). Alternatively, one may exploit the correspondence described below.

There is a bijective correspondence between QS convolution cocycles and their opposite counterparts. It may be implemented either by the time-reversal operation or by passage to the opposite coalgebra. To formulate this precisely we need to introduce some further notation. For each $t \geq 0$ define the following time reversal operator on $L^2(\mathbb{R}_+; \mathsf{k})$:

$$(r_t(f))(u) = \begin{cases} f(u-t) & \text{if } u \leq t \\ f(u) & \text{if } u > t \end{cases}, \qquad f \in L^2(\mathbb{R}_+; \mathsf{k}),\ u \geq 0.$$

This in turn may be second-quantised to the unitary selfadjoint operator $R_t \in B(\mathcal{F})$ whose action on exponential vectors is given by

$$R_t(\varepsilon(f)) = \varepsilon(r_t f).$$

The latter provides, by conjugation, time reversal on the level of operators in $\mathcal{O}(\mathcal{E}_D)$:

$$\rho_t(Z) = R_t Z R_t, \ \ Z \in \mathcal{O}(\mathcal{E}_D).$$

Recall also that if $\mathcal{C}$ is a coalgebra with coproduct Δ, the *opposite coalgebra* $\mathcal{C}^{\mathrm{opp}}$ is the same vector space as $\mathcal{C}$ equipped with the same counit and with the coproduct $\tau \circ \Delta$, where τ denotes the tensor flip on $\mathcal{C} \odot \mathcal{C}$.

Proposition 3.8.6. *Let $l \in \mathbb{P}(\mathcal{C}; \mathcal{E}_D)$. Then the following are equivalent :*

(i) *l is an opposite QS convolution cocycle on $\mathcal{C}$;*

(ii) *the process $\widetilde{l} \in \mathbb{P}(\mathcal{C}^{opp}; \mathcal{E}_D)$ given by*

$$\widetilde{l}_t = l_t, \quad t \geq 0,$$

is a QS convolution cocycle on $\mathcal{C}^{opp}$;

(iii) *the process $\widetilde{l} \in \mathbb{P}(\mathcal{C}; \mathcal{E}_D)$ defined by*

$$\widetilde{l}_t = \rho_t \circ l_t, \quad t \geq 0,$$

is a QS convolution cocycle on $\mathcal{C}$.

Proof. All the equivalences follow directly from the relevant semigroup decomposition formulas (3.2.3) and (3.8.2). $\square$

In particular time-reversal (or passage to the opposite coalgebra) exchanges l^{φ} with ${}^{\varphi}l$.

Finally note that in [LS$_1$] we actually worked with opposite cocycles (although it was not clarified there).

Chapter 4

C^*-algebraic case

This chapter is concerned with quantum stochastic convolution cocycles on an operator space coalgebra, in the second part specialised to a C^*-hyperbialgebra or a C^*-bialgebra. Many results in this part mirror the ones obtained in the purely algebraic case. Technical conditions under which theorems hold are however usually different, and although some proofs are similar, it is not possible to conduct them precisely so that the argument is valid for both cases.

The operator space coalgebraic structure, defined by analogy with the purely algebraic case, allows for defining the convolution product. In the topological context it becomes important that the convolution may be transformed via the so-called R-map into a composition operation preserving all the relevant continuity properties. This corresponds to the procedure, familiar from classical probability, of transforming the convolution of measures into the composition of Markov operators. On the level of cocycles the R-map transforms a QS convolution cocycle, with its associated convolution semigroups, into a standard QS cocycle, with its associated semigroups.

Operator-space theoretic QS convolution cocycles, as in the purely algebraic case, are obtained by solving coalgebraic QS differential equations. This time however to prove the existence and uniqueness theorems for solutions we need to assume complete boundedeness of coefficients (in [LS_3] it is shown that in fact it is enough to assume that the coefficient has completely bounded columns). The methods used here extend those of [LW_3], allowing nontrivial initial conditions. As in general there is no guarantee that the solutions of a given coalgebraic QS differential equation will be completely bounded, in this topological context we need to distinguish the class of weak QS convolution cocycles. These satisfy the cocycle relation only on the level of relevant matrix elements.

A converse of the fact that each solution of a coalgebraic QS differential equation is a weak QS convolution cocycle may be obtained when the initial object is a C^*-hyperbialgebra, so that one can exploit the order structure. Using the known results from the theory of standard QS cocycles ([LiP],

[LW$_2$]) and the R-map described in the previous paragraph we prove that every Markov-regular, completely positive and contractive QS convolution cocycle on a C^*-hyperbialgebra satisfies a coalgebraic QS differential equation. We also characterise the general form of stochastic generators yielding such cocycles. This leads further to the characterisation of stochastic generators of *-homomorphic QS convolution cocycles on a C^*-bialgebra A in terms of structure maps on A (which can again be equivalently described via Schürmann triples). The interesting feature here is that 'the algebra determines the analysis' - every structure map on A is necessarily completely bounded, and even inner. The proof of this fact is based on versions of well-known results of S. Sakai, J.R. Ringrose and E. Christensen on continuity and innerness properties of derivations for the case of (π_1, π_2)-derivations (the proofs are provided in Appendix A). The innerness of structure maps on A may be viewed as a noncommutative counterpart to the fact that every classical Lévy process on a topological group which has a bounded generator must be a compound Poisson process.

It is possible to axiomatise quantum Lévy processes on C^*-bialgebras, either using solely the concept of distributions, or exploiting the language of Arveson's theory of product systems. Accordingly, we propose definitions of a weak quantum Lévy process and a product system quantum Lévy process. For each of them a topological version of the Schürmann Reconstruction Theorem remains valid.

The explicit characterisations of the stochastic generators enable us to prove two types of dilation theorems for completely positive and contractive QS convolution cocycles on a C^*-bialgebra, corresponding to the dilations obtained for standard QS cocycles in [GLSW] and [GLW].

After the general theory has been presented, we describe the basic examples of commutative, cocommutative and *genuinely quantum* C^*-bialgebras. Different perspectives on QS convolution cocycles or their stochastic generators in each of the cases is offered. In particular in the context of full compact quantum groups purely algebraic and operator space theoretic cocycles are shown

to coexist on the same underlying space. This supports the view (explicitly described in the expository paper [LS_2]) that despite the obvious differences both categories of QS convolution cocycles may nevertheless usefully be seen from a common vantage point.

It would be interesting to obtain extensions of the results of this chapter to multiplier C^*-bialgebras (locally compact quantum semigroups) - a short explanation of the technical difficulties related to such a project is provided in the last section.

Finally we would like to stress that, in contrast to [LS_4], all operator space coalgebras, C^*-hyperbialgebras and C^*-bialgebras are assumed to be concrete. For the latter two we mean by that explicitly that they are represented in a faithful and nondegenerate way on a Hilbert space.

4.1 Operator space coalgebras and convolution semigroups

In this section we present the definition of an operator space coalgebra, establish basic facts concerning the convolution product provided by the coalgebraic structure and define the R-map facilitating the traffic between the convolution and composition operations.

Definition 4.1.1. An operator space C is an operator space (OS) coalgebra if there are completely contractive maps $\Delta : \mathsf{C} \to \mathsf{C} \otimes \mathsf{C}$ and $\epsilon : \mathsf{C} \to \mathbb{C}$, called the *coproduct* and *counit* respectively, enjoying the coassociativity and the counit property, namely

$$(\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta, \tag{4.1.1}$$

$$(\text{id} \otimes \epsilon) \circ \Delta = (\epsilon \otimes \text{id}) \circ \Delta = \text{id}. \tag{4.1.2}$$

The formula (3.1.3) defines again a map Δ_n, and this time $\Delta_n : \mathsf{C}^{\otimes n} \to \mathsf{C}^{\otimes n+1}$ - recall that $\Delta_0 := \text{id}_{\mathsf{C}}$. As we no longer can expect that $\Delta(a)$ ($a \in \mathsf{C}$)

is a finite sum of simple tensors, we generally avoid Sweedler notation in this chapter.

Definition 4.1.2. A unital C^*-algebra A is called a C^*-hyperbialgebra, if it is an OS coalgebra, the counit is a character (unital multiplicative functional) and the coproduct is unital and completely positive. A C^*-hyperbialgebra is called a C^*-bialgebra if its coproduct is multiplicative (so *-homomorphic).

Remark 4.1.3. One can also introduce an apparently natural category of operator system coalgebras, but it will not be of much use here. Some authors reserve the name C^*-bialgebra for a not-necessarily-unital C^*-algebra A with the coproduct taking values in $M(\mathsf{A} \otimes \mathsf{A})$ (the multiplier algebra of $\mathsf{A} \otimes \mathsf{A}$); this is relevant for the considerations in Section 4.9.

The motivating examples of C^*-bialgebras and C^*-hyperbialgebras come respectively from the theory of compact quantum groups ([Wor$_1$], [Wor$_2$]) and hypergroups ([ChV]).

Let us introduce a modification of the convolution product $\star$ of Chapter 2. If C is an OS coalgebra, $\mathsf{V}_1, \mathsf{V}_2$ - operator spaces and $\varphi_1 \in CB(\mathsf{C}; \mathsf{V}_1)$, $\varphi_2 \in CB(\mathsf{C}; \mathsf{V}_2)$, define

$$\varphi_1 \star \varphi_2 = (\varphi_1 \otimes \varphi_2) \circ \Delta : \mathsf{C} \to \mathsf{V}_1 \otimes \mathsf{V}_2.$$

It is easily seen that $\star$ is associative and enjoys submultiplicativity and unital properties:

$$\varphi_1 \star \varphi_2 \star \varphi_3 = (\varphi_1 \otimes \varphi_2 \otimes \varphi_3) \circ \Delta_2 \tag{4.1.3}$$

$$\|\varphi_1 \star \varphi_2\|_{\text{cb}} \leq \|\varphi_1\|_{\text{cb}} \|\varphi_2\|_{\text{cb}}, \text{ and}$$

$$\epsilon \star \varphi = \varphi = \varphi \star \epsilon. \tag{4.1.4}$$

In particular, $(\mathsf{C}^*, \star)$ is a unital Banach algebra. The following observation will be used further: if V is an operator space, $\varphi \in CB(\mathsf{C}; \mathsf{V})$ and $n \in \mathbb{N}$, then

$$\varphi^{\star n} = \varphi^{\otimes n} \circ \Delta_{n-1}. \tag{4.1.5}$$

Define also $\varphi^{\star 0} := \epsilon$, which is consistent with (4.1.4).

Definition 4.1.4. A convolution semigroup of functionals (CSF, for short) on an OS coalgebra C is a family $\{\kappa_s : s \geq 0\}$ of bounded linear functionals on C satisfying the following conditions:

$$\kappa_{s+t} = \kappa_s \star \kappa_t, \quad s, t \geq 0, \tag{4.1.6}$$

$$\kappa_0 = \epsilon. \tag{4.1.7}$$

If $\|\kappa_s - \epsilon\| \to 0$ as $s \to 0^+$ then $\{\kappa_s : s \geq 0\}$ is said to be norm continuous (it is then abbreviated to CCSF).

The notions of cocommutative coalgebras and idempotent functionals introduced in Section 3.1 have obvious counterparts in this context.

Convolution and the R-map

To analyse existence and properties of generators of CCSFs, and also for the investigations in Section 4.2 we need to describe the properties of a certain map 'linearising' the convolution.

Given an operator space coalgebra C, each operator space V determines maps

$$R_{\mathsf{V}} : CB(\mathsf{C}; \mathsf{V}) \to CB(\mathsf{C}; \mathsf{C} \otimes \mathsf{V}), \quad \varphi \mapsto (\mathrm{id}_{\mathsf{C}} \otimes \varphi) \circ \Delta = \mathrm{id}_{\mathsf{C}} \star \varphi;$$
$$E_{\mathsf{V}} : CB(\mathsf{C}; \mathsf{C} \otimes \mathsf{V}) \to CB(\mathsf{C}; \mathsf{V}), \quad \phi \mapsto (\epsilon \otimes \mathrm{id}_{\mathsf{V}}) \circ \phi.$$

Thus the action of R_{V} is given by the convolution with the identity map on C, putting the argument on the right, and the action of E_{V} is given by the composition with the tensor-extension of the counit. Symbols $R_{\mathbb{C}}$ and $E_{\mathbb{C}}$ will be respectively abbreviated to R and E.

The following proposition is a straightforward consequence of the definitions of R and E.

Proposition 4.1.5. *Let C be an OS coalgebra, and let V_1, V_2, V be operator spaces.*

(a) *R_{V} and E_{V} are complete isometries satisfying*

$$E_{\mathsf{V}} \circ R_{\mathsf{V}} = id_{CB(\mathsf{C};\mathsf{V})}.$$

(b) *If $\varphi_1 \in CB(\mathsf{C};\mathsf{V}_1)$ and $\varphi_2 \in CB(\mathsf{C};\mathsf{V}_2)$ then*

$$R_{\mathsf{V}_1 \otimes \mathsf{V}_2}(\varphi_1 \star \varphi_2) = (R_{\mathsf{V}_1}\varphi_1 \otimes id_{\mathsf{V}_2}) \circ R_{\mathsf{V}_2}\varphi_2.$$

Write $CB^{\Delta}(\mathsf{C};\mathsf{C} \otimes \mathsf{V})$ for $\operatorname{Ran} R_{\mathsf{V}}$.

Corollary 4.1.6. *For each operator space V, R_{V} is a complete isometry of operator spaces*

$$CB(\mathsf{C};\mathsf{V}) \cong CB^{\Delta}(\mathsf{C};\mathsf{C} \otimes \mathsf{V}).$$

Moreover, R is an isometric isomorphism of unital Banach algebras

$$(\mathsf{C}^*, \star) \cong \big(CB^{\Delta}(\mathsf{C}), \circ\big).$$

A further interesting consequence is the following

Corollary 4.1.7. *In $CB^{\Delta}(\mathsf{C};\mathsf{C} \otimes M_n)$,*

$$\|\phi\|_{\mathrm{cb}} = \|\phi^{(n)}\|.$$

Proof. Let $\phi \in CB^{\Delta}(\mathsf{C};\mathsf{C} \otimes M_n)$, say $\phi = R_{M_n}\varphi$ for some $\varphi \in CB(\mathsf{C};M_n)$. Then

$$\|\phi\|_{\mathrm{cb}} = \|\varphi\|_{\mathrm{cb}} = \|\varphi^{(n)}\| = \|(\epsilon \otimes \mathrm{id}_{M_n(\mathsf{C})})\phi^{(n)}\| \leq \|\phi^{(n)}\|;$$

the result follows. $\square$

In particular, in $CB^{\Delta}(\mathsf{C})$ the completely bounded norm coincides with the bounded operator norm. As a result $CB^{\Delta}(\mathsf{C})$ is a closed subspace of $B(\mathsf{C})$. The next proposition collects the structure-preserving properties of the map R_{V} and E_{V} under a number of relevant assumptions on C and V.

Proposition 4.1.8. *Let C be an OS coalgebra and V an operator space, let $\varphi \in CB(\mathsf{C};\mathsf{V})$ and $\phi = R_{\mathsf{V}}\varphi \in CB(\mathsf{C};\mathsf{C}\otimes\mathsf{V})$.*

(a) *ϕ is completely contractive if and only if φ is.*

(b) *If C is a C^*-hyperbialgebra and V is an operator system then ϕ is real (respectively, completely positive, or unital) if and only if φ is.*

(c) *If C is a C^*-bialgebra and V is a C^*-algebra then ϕ is multiplicative if and only if φ is.*

The following proposition paves the way for analysing generators of convolution semigroups of functionals on OS coalgebras.

Proposition 4.1.9. *Let C be an operator space coalgebra. The map $\kappa := \{\kappa_t : t \geq 0\} \mapsto P := \{(R\kappa_t)_{t\geq 0}\}$ is a bijection from the set of CSFs on C to the set of one-parameter semigroups in $CB^{\Delta}(\mathsf{C})$. Moreover, the conditions in (a) below are equivalent, and so are the conditions in (b):*

(a) (i) $\lim_{t\to 0}\kappa_t(a) = \epsilon(a)$ *for all $a \in \mathsf{C}$;*

(ii) *P is a c_0-semigroup on C.*

(b) (i) *κ is norm continuous in t;*

(ii) *P is norm continuous in t;*

(iii) *P is* cb-*norm continuous in t;*

(iv) *the generator of P is completely bounded.*

Proof. The first part follows from Corollary 4.1.6.

(a) Since $\epsilon \circ P_t = \kappa_t$, (ii) implies (i). Suppose therefore that (i) holds. Then, for any $\lambda \in \mathsf{C}^*$,

$$\lambda \circ P_t = \kappa_t \circ (\lambda \otimes \mathrm{id}_{\mathsf{C}}) \circ \Delta$$

and

$$\epsilon \circ (\lambda \otimes \mathrm{id}_{\mathsf{C}}) \circ \Delta = \lambda,$$

so $P_t a$ converges to a weakly as $t \to 0^+$ for all $a \in \mathsf{C}$. But this implies that P is strongly continuous ([Dav], Proposition 1.23) and thus a c_0-semigroup.

(b) By Corollary 4.1.7

$$\|P_t - \mathrm{id}_{\mathsf{C}}\|_{\mathrm{cb}} = \|\kappa_t - \epsilon\| = \|P_t - \mathrm{id}_{\mathsf{C}}\|,$$

and so (b) follows. □

Thus each norm-continuous CSF $\{\kappa_t : t \geq 0\}$ on C has a *generator*:

$$\gamma := \lim_{t \to 0^+} \frac{(\kappa_t - \epsilon)}{t}$$

from which the CSF may be recovered:

$$\kappa_t = \exp_\star t\gamma := \sum_{n \geq 0} (n!)^{-1} t^n \gamma^{\star n}$$

$(t \geq 0)$, where $\gamma_0 := \epsilon$.

The corresponding one-parameter semigroup on C has a completely bounded generator: $R\kappa_t = \mathrm{e}^{t\tau}$, $t \geq 0$, where $\tau = R\gamma \in CB(\mathsf{C})$. Remarks after Definition 3.1.5 apply also here.

4.2 QS convolution cocycles and standard QS cocycles

Here the main object of the thesis is defined in the operator space theoretic context, and the connection with the standard theory of QS cocycles is made.

QS convolution cocycles on OS coalgebras

Definition 4.2.1. A *quantum stochastic convolution cocycle* (on C with domain $\mathcal{E}$) is a process $l \in \mathbb{P}_{\mathrm{cb}}(\mathsf{C}; \mathcal{E})$ such that, for all $s, t \geq 0$,

$$l_{s+t} = l_s \star (\sigma_s \circ l_t) \tag{4.2.1}$$

and for all $a \in \mathsf{C}$

$$l_0(a) = \epsilon(a) I_{\mathcal{F}}. \tag{4.2.2}$$

Family of all such QS convolution cocycles is denoted by $\mathbb{QSCC}(\mathsf{C}; \mathcal{E})$; the notation adorned with subscripts and superscript † according to our convention. Remark 3.2.2 remains valid, and QS convolution cocycles on C have associated CSFs, as the following lemma shows.

Lemma 4.2.2. *Let $l \in \mathbb{P}_{\mathrm{cb}}(\mathsf{C}; \mathcal{E})$ be a convolution increment process. For each $f, g \in \mathbb{S}$,*

$$\mathrm{e}^{-\langle f_{[0,t)}, g_{[0,t)} \rangle} \left\langle \varepsilon(f_{[0,t)}), l_t(a) \varepsilon(g_{[0,t)}) \right\rangle = \left(\lambda_{t_1 - t_0}^{f(t_0), g(t_0)} \star \cdots \star \lambda_{t_n - t_{n-1}}^{f(t_{n-1}), g(t_{n-1})} \right)(a) \tag{4.2.3}$$

where $0 = t_0 \leq t_1 \leq \cdots \leq t_n = t$ contains the discontinuities of $f_{[0,t)}$ and $g_{[0,t)}$, and

$$\lambda_t^{c,d} := \mathrm{e}^{-t\langle c,d \rangle} \left\langle \varepsilon(c_{[0,t)}), l_t(\,\cdot\,) \varepsilon(d_{[0,t)}) \right\rangle, \quad c, d \in \mathsf{k}. \tag{4.2.4}$$

Proof. As for purely algebraic cocycles, the identity (4.2.3) results from repeated application of the cocycle relation (4.2.1). ∎

In the topological case there is a need to distinguish QS convolution cocycles from the class of processes satisfying (4.2.3). Recall the notion of weakly bounded processes introduced in Section (2.3).

Definition 4.2.3. A process $l \in \mathbb{P}_{\mathrm{wb}}(\mathsf{C}; \mathcal{E})$ is a weak convolution increment process if it satisfies formula (4.2.3); it is a weak QS convolution cocycle if in addition $l_0(a) = \epsilon(a) I_{\mathcal{F}}$ for all $a \in \mathsf{C}$.

Remark 4.2.4. Note that weak boundedness of l guarantees boundedness (so also complete boundedness) of all functionals $\lambda^{c,d}$ and the right hand side of formula (4.2.3) makes sense. Lemma 4.2.2 says that each QS convolution cocycle is a weak QS convolution cocycle. It is easy to observe that if a weak QS convolution cocycle is completely bounded, it is a QS convolution cocycle.

Corollary 4.2.5. *Let $l \in \mathbb{P}_{\mathrm{wb}}(\mathsf{C}; \mathcal{E})$ be a weak QS convolution cocycle, $c, d \in \mathsf{k}$. Then* (4.2.4) *defines a convolution semigroup of functionals on C : $(\lambda_t^{c,d})_{t \geq 0}$.*

Definition 4.2.6. Semigroups defined by (4.2.4) (for all pairs $c, d \in \mathsf{k}$) are called associated convolution semigroups of the cocycle l. The semigroup $\lambda^{0,0}$ is called the Markov semigroup of the cocycle l.

It is clear that two weak QS convolution cocycles with identical associated convolution semigroups are equal. For further analysis we need one more definition.

Definition 4.2.7. A weak QS convolution cocycle is Markov-regular if its Markov semigroup is norm-continuous.

Standard QS cocycles

Standard QS cocycles have been analysed in numerous papers. Here we follow the point of view of [Lin] (see also [LW_1] and references therein); the terminology is however different, as we intend to treat completely bounded processes as the objects of primary interest. None of the results in this section is new, they are recalled here as they will be used to prove many important properties of QS convolution cocycles.

Let $\mathsf{V} \subset B(\mathsf{h})$ be a concrete operator space. Recall the matrix spaces introduced in Section 2.2. Whenever $k \in \mathbb{P}_{\mathrm{cb}}(\mathsf{V};\mathsf{V},\mathcal{E})$, $s,t \geq 0$, we obtain the map $\widehat{\sigma}_s \circ k_t : \mathsf{V} \mapsto M_{\mathcal{F}_{[s,\infty)}}(\mathsf{V})$, where $\widehat{\sigma}_s$ is a natural extension of the shift endomorphism to $M_{\mathcal{F}}(\mathsf{V})$ and k_s may be equivalently viewed as a map from V to $M_{\mathcal{F}_{[0,s)}}(\mathsf{V})$. Denote by $\widehat{k}_s$ the matricial extension of k_s to $M_{\mathcal{F}_{[s,\infty)}}(\mathsf{V})$. The functorial property (2.2.1) and the tensor factorisation (2.3.1) of the Fock space imply that the following definition is consistent.

Definition 4.2.8. A process $k \in \mathbb{P}_{\mathrm{cb}}(\mathsf{V};\mathsf{V},\mathcal{E})$ is a standard QS cocycle (on V with domain $\mathcal{E}$) if

$$k_{s+t} = \widehat{k}_s \circ \widehat{\sigma}_s \circ k_t, \tag{4.2.5}$$

and for all $v \in \mathsf{V}$

$$k_0(v) = v \otimes I_{\mathcal{F}}. \tag{4.2.6}$$

The associated semigroups of a standard QS cocycle are semigroups acting on V, as the next lemma shows:

Lemma 4.2.9. *Let $k \in \mathbb{P}_{\mathrm{cb}}(\mathsf{V};\mathsf{V},\mathcal{F})$ be a standard QS cocycle. For each $f,g \in \mathbb{S}$,*

(4.2.7)
$$\mathrm{e}^{-\langle f_{[0,t)}, g_{[0,t)}\rangle} \left\langle \varepsilon(f_{[0,t)}), k_t(a)\varepsilon(g_{[0,t)})\right\rangle = \left(P^{f(t_0),g(t_0)}_{t_1-t_0} \circ \cdots \circ P^{f(t_{n-1}),g(t_{n-1})}_{t_n-t_{n-1}}\right)(a)$$

where $0 = t_0 \leq t_1 \leq \cdots \leq t_n = t$ contains the discontinuities of $f_{[0,t)}$ and $g_{[0,t)}$, and

$$P^{c,d}_t := \mathrm{e}^{-t\langle c,d\rangle} \left\langle \varepsilon(c_{[0,t)}), k_t(\,\cdot\,)\varepsilon(d_{[0,t)})\right\rangle \quad c,d \in \mathsf{k}. \tag{4.2.8}$$

Definition 4.2.10. A process $k \in \mathbb{P}_{\mathrm{wb}}(\mathsf{V};\mathsf{V},\mathcal{E})$ is a weak standard QS cocycle if it satisfies formulas (4.2.7) and (4.2.6).

Remark 4.2.11. Lemma 4.2.9 says that each standard QS cocycle is a weak standard QS cocycle; again, it is easy to observe that if a weak standard QS

cocycle is completely bounded, it is a standard QS cocycle.

Corollary 4.2.12. *Let $k \in \mathbb{P}_{\mathrm{wb}}(\mathsf{V}; \mathsf{V}, \mathcal{E})$ be a weak QS cocycle, $c, d \in \mathsf{k}$. Then* (4.2.8) *defines a semigroup acting on V: $(P_t^{c,d})_{t \geq 0}$.*

Definition 4.2.13. Semigroups defined by (4.2.8) (for all pairs $c, d \in \mathsf{k}$) are called associated semigroups of the cocycle k. The semigroup $P^{0,0}$ is called the Markov semigroup of the cocycle k.

It is clear that two weak standard QS cocycles with identical associated semigroups are equal.

Definition 4.2.14. A weak standard QS cocycle is Markov-regular if its Markov semigroup is norm-continuous.

The crucial result here, connecting the two types of cocycles introduced above is the following consequence of Corollary 4.1.6.

Fact 4.2.15. *Let $l \in \mathbb{P}_{\mathrm{wb}}(\mathsf{C}; \mathcal{E})$, $k \in \mathbb{P}_{\mathrm{wb}}(\mathsf{C}; \mathsf{C}, \mathcal{E})$ and assume that for all $f, g \in \mathbb{S}$, $t \geq 0$*

$$E^{\varepsilon(f)} k_{t,\varepsilon(g)} = R\left(E^{\varepsilon(f)} l_{t,\varepsilon(g)}\right).$$

Then l is completely bounded if and only if k is, l is a weak QS convolution cocycle if and only if k is a weak standard QS cocycle, and in the latter case l is Markov-regular if and only if k is.

Proof. The first fact follows directly from the definition of map R; observe that if either l or k (so in fact both) is completely bounded, then, for each $t \geq 0$, $k_t = R_{B(\mathcal{F})} l_t$. The second fact follows from Corollary 4.1.6, which gives for all $f_1, f_2, g_1, g_2 \in \mathbb{S}$, $t \geq 0$

$$\begin{aligned} R\left(\left(E^{\varepsilon(f_1)} l_{t,\varepsilon(g_1)}\right) \star \left(E^{\varepsilon(f_2)} l_{t,\varepsilon(g_2)}\right)\right) \\ = \left(E^{\varepsilon(f_1)} k_{t,\varepsilon(g_1)}\right) \circ \left(E^{\varepsilon(f_2)} k_{t,\varepsilon(g_2)}\right). \end{aligned}$$

$\square$

4.3 QS differential equations in operator space context

As we have already seen in Section 3.3, to obtain QS convolution cocycles as solutions of QS differential equations it is necessary to consider equations with nontrivial initial conditions. Therefore in this section we extend the results of [LW$_3$] to such situations. As usual in this chapter, the main stress will be put on completely bounded coefficients (called also stochastic generators).

Let V, W ($\mathsf{W} \subset B(\mathsf{h})$) be operator spaces and assume that $\phi \in CB(\mathsf{V}; M_{\widehat{\mathsf{k}}}(\mathsf{V}))$, $\theta \in CB(\mathsf{V}; \mathsf{W})$. By a QS differential equation with the coefficient ϕ and the initial condition θ is understood the equation

$$dk_t = \widehat{k}_t \circ \phi \, d\Lambda_t, \quad k_0 = \iota \circ \theta. \tag{4.3.1}$$

A process $k \in \mathbb{P}(\mathsf{V}; \mathsf{W}, \mathcal{E})$ is a *weak solution* of the equation (4.3.1) if for all $\xi, \eta \in \mathsf{h}$, $f, g \in \mathbb{S}$, $t \geq 0$, $v \in \mathsf{V}$

(4.3.2)
$$\left\langle \xi\varepsilon(f), \left(k_t(v) - \theta(v) \otimes I_{\mathcal{F}}\right)\eta\varepsilon(g)\right\rangle = \int_0^t \left\langle \xi\varepsilon(f), k_s\left(E^{\hat{f}(s)}\phi(v)E_{\hat{g}(s)}\right)\eta\varepsilon(g)\right\rangle ds.$$

The definition of a strong solution requires more care - we need to explain how to define the process that is actually QS integrated and whose matrix elements are represented by the right hand side of (4.3.2). Assume that a process $k \in \mathbb{P}(\mathsf{V}; \mathsf{W}, \mathcal{E})$ has completely bounded columns. For each $t \geq 0$, $f \in \mathbb{S}$, define

$$K_{t,\varepsilon(f)} = \tau \circ (k_{t,\varepsilon(f)})^{(\widehat{\mathsf{k}})} \circ \phi : \mathsf{V} \to C_{\mathcal{F}}\left(M_{\widehat{\mathsf{k}}}(\mathsf{W})\right)$$

(τ denotes here a map implemented by the tensor flip $\mathsf{h} \otimes \mathcal{F} \otimes \widehat{\mathsf{k}} \mapsto \mathsf{h} \otimes \widehat{\mathsf{k}} \otimes \mathcal{F}$ - in fact this is the same flip which is explicitly present in coalgebraic QS differential equations (3.3.1) and (4.4.1)). Let the process $K \in \mathbb{P}(\mathsf{V}; M_{\widehat{\mathsf{k}}}(\mathsf{W}), \mathcal{E})$ be defined

by

$$K_t(v)(\xi \otimes \zeta \otimes \varepsilon(f)) = K_{t,\varepsilon(f)}(v)(\xi \otimes \zeta) \tag{4.3.3}$$

($t \geq 0, \xi \in \mathsf{h}, \zeta \in \widehat{\mathsf{k}}, f \in \mathbb{S}$, $v \in \mathsf{V}$).

We say that a process $k \in \mathbb{P}(\mathsf{V}; \mathsf{W}, \mathcal{E})$ is a *strong solution* of the equation (4.3.1) if it is a weak solution, it has completely bounded columns and the process K introduced above is locally square integrable. In this case the First Fundamental Formula implies that for each $v \in \mathsf{V}$

$$k_t(v) = \theta(v) \otimes I_{\mathcal{F}} + \int_0^t K_s(v)\, d\Lambda_s.$$

Uniqueness of the solution

Proposition 4.3.1. *The QS differential equation* (4.3.1) *has at most one weakly regular weak solution.*

Proof. Let $k \in \mathbb{P}_{\mathrm{wr}}(\mathsf{V}; \mathsf{W}, \mathcal{E})$ be the difference of two weak solutions of (4.3.1), and let $v \in \mathsf{V}$, $\xi, \eta \in \mathsf{h}$, $f, g \in \mathbb{S}_D$ and $T \geq 0$. Then by iteration

$$\langle \xi\varepsilon(f), k_t(v)\eta\varepsilon(g)\rangle = \int_{\Delta_n[0,t]} \Big\langle \xi\varepsilon(f), k_{s_1}\Big(\phi^{\hat{f}(s_1)}_{\hat{g}(s_1)} \circ \cdots \circ \phi^{\hat{f}(s_n)}_{\hat{g}(s_n)}(v)\Big)\eta\varepsilon(g)\Big\rangle\, d\boldsymbol{s}$$

for each $n \in \mathbb{N}$ and $t \in [0, T]$, where for $\zeta_1, \zeta_2 \in \widehat{\mathsf{k}}$, $w \in \mathsf{V}$

$$\phi^{\zeta_1}_{\zeta_2}(w) := E^{\zeta_1}\phi(w)E_{\zeta_2}.$$

Now weak regularity of k allows us to claim that the integrand is bounded by

$$C_{f,g,T}\Big(\max\{\|\hat{d}\|^2 : d \in \operatorname{Ran} f \cup \operatorname{Ran} g\}\|\phi\|_{\mathrm{cb}}\Big)^n \|v\|\|\xi\|\|\eta\|,$$

where $C_{f,g,T} > 0$ is a certain constant. The result follows. $\square$

It is clear from the proof above that for the uniqueness of the solution one does not need to assume any continuity properties of the initial condition.

Existence of the solution

The solution of the equation (4.3.1) is constructed by means of iterated QS integrals. Recall the notations of Section 2.4 and the formal inclusion $B(\mathsf{h}) \subset \mathcal{O}(E)$ (for E - dense subspace of h). Let $\upsilon^{\theta,\phi}$ be the linear map $\mathsf{V} \to \mathscr{S}_{\mathsf{h},\mathsf{k}}$ defined by $\upsilon^{\theta,\phi}(v)_n = \upsilon^{\theta,\phi}_n(v)$ $(n \in \mathbb{N}_0)$ where

$$\upsilon^{\theta,\phi}_n = \theta_n \circ \phi_n,$$

the maps $\phi_n : \mathsf{V} \to M_{\widehat{\mathsf{k}}^{\otimes n}}(\mathsf{V})$ and $\theta_n : M_{\widehat{\mathsf{k}}^{\otimes n}}(\mathsf{V}) \to M_{\widehat{\mathsf{k}}^{\otimes n}}(\mathsf{W})$ are defined (recursively) by $\phi_0 = \mathrm{id}_{\mathsf{V}}$, $\theta_0 = \theta$, $\phi_1 = \phi$, $\theta_1 = \theta^{(\widehat{\mathsf{k}})}$, and, for $n \geq 2$, $\phi_n = \phi^{(\widehat{\mathsf{k}}^{\otimes (n-1)})} \circ \phi_{n-1}$, $\theta_n = \theta^{(\widehat{\mathsf{k}}^{\otimes n})}$. Let $\widetilde{\upsilon}^{\theta,\phi} : \mathsf{V} \to \mathscr{S}_{\mathsf{h},\mathsf{k}}$ be given by $(n \in \mathbb{N}_0)$

$$\widetilde{\upsilon}^{\theta,\phi}_n = \tau_n \circ \upsilon^{\theta,\phi}_n,$$

where $\tau_n : B(\mathsf{h} \otimes \widehat{\mathsf{k}}^{\otimes n}) \to B(\mathsf{h} \otimes \widehat{\mathsf{k}}^{\otimes n})$ is the tensor flip reversing the order of the copies of $\widehat{\mathsf{k}}$.

The reader will recognise here relevant continuous extensions of the maps introduced in Section 3.3.

Lemma 4.3.2. *For any $\phi \in CB(\mathsf{V}; M_{\widehat{\mathsf{k}}}(\mathsf{V}))$ and $\theta \in CB(\mathsf{V}; \mathsf{W})$, the maps $\upsilon^{\theta,\phi}$, $\widetilde{\upsilon}^{\theta,\phi}$ take values in $\mathcal{G}^{\dagger}_{\mathsf{h},\mathsf{k}}$.*

Proof. The result follows from properties of the liftings of completely bounded maps to matrix spaces, implying the estimates $(n \in \mathbb{N}_0)$

$$\|\phi_n\| \leq \|\phi\|^n_{\mathrm{cb}}, \quad \|\theta_n\| \leq \|\theta\|_{\mathrm{cb}}, \quad \|\upsilon^{\theta,\phi}_n\| \leq \|\phi\|^n_{\mathrm{cb}} \|\theta\|_{\mathrm{cb}}.$$

□

Theorem 4.3.3. *Let $\phi \in CB(\mathsf{V}; M_{\widehat{\mathsf{k}}}(\mathsf{V}))$ and $\theta \in CB(\mathsf{V}; \mathsf{W})$. Then the process $k := \Lambda \circ \widetilde{\upsilon}^{\theta,\phi}$ is a strong solution of the QS differential equation* (4.3.1).

Proof. Write $\upsilon := \upsilon^{\theta,\phi}, \widetilde{\upsilon} := \widetilde{\upsilon}^{\theta,\phi}$ and prove first the following equality:

$$\upsilon_n(E^{\zeta}\phi(v)E_{\chi}) = E^{\zeta}\upsilon_{n+1}(v)E_{\chi}, \tag{4.3.4}$$

where $\zeta, \chi \in \widehat{\mathsf{k}}$, $v \in \mathsf{V}$, $n \in \mathbb{N}$ (note that on the right hand side Dirac operators are applied to the last copy of $\widehat{\mathsf{k}}$ in the tensor product). It is enough to show that

$$\phi_n(E^\zeta \phi(v) E_\chi) = E^\zeta \phi_{n+1}(v) E_\chi,$$

and for that it is enough to compare matrix elements of both sides, as all the maps involved are (completely) bounded. Equality of the matrix elements follows from the inductively proved formula:

$$E^{\zeta_1} \dots E^{\zeta_n} \phi_n(v) E_{\chi_n} \dots E_{\chi_1} = E^{\zeta_1} \left(\phi \left(\cdots \left(E^{\zeta_n} \phi(v) E_{\chi_n} \right) \cdots \right) \right) E_{\chi_1},$$

valid for all $n \in \mathbb{N}$, $v \in \mathsf{V}$, $\zeta_1, \dots, \zeta_n, \chi_1, \dots, \chi_n \in \widehat{\mathsf{k}}$.

Note that there is a version of the formula (4.3.4) for the map $\widetilde{v}$:

$$\widetilde{v}_n(E^\zeta \phi(v) E_\chi) = E^\zeta \widetilde{v}_{n+1}(v) E_\chi; \tag{4.3.5}$$

the only difference being that Dirac operators on the right hand side are now applied to the first copy of $\widehat{\mathsf{k}}$ in the tensor product. With (4.3.5) in hand we are ready to prove that k satisfies (4.3.1) weakly. Choose $f, g \in \mathbb{S}$, $v \in \mathsf{V}$, $\xi, \eta \in \mathsf{h}$. In Guichardet notation, (2.4.10) implies

$$\begin{aligned}
&\int_0^t ds \left\langle \xi\varepsilon(f), k_s \left(E^{\hat{f}(s)} \phi(v) E_{\hat{g}(s)} \right) \eta\varepsilon(g) \right\rangle \\
&= \int_0^t ds \int_{\Gamma_{[0,s]}} d\tau \left\langle \xi \pi_{\hat{f}}(\tau), \big(\widetilde{v}_{\#\tau}(E^{\hat{f}(s)} \phi(v) E_{\hat{g}(s)})\big) \eta \pi_{\hat{g}}(\tau) \right\rangle \langle \varepsilon(f), \varepsilon(g) \rangle \\
&= \int_0^t ds \int_{\Gamma_{[0,s]}} d\tau \left\langle \xi \pi_{\hat{f}}(\tau \cup s), \big(\widetilde{v}_{\#(\tau\cup s)}(v)\big) \eta \pi_{\hat{g}}(\tau \cup s) \right\rangle \langle \varepsilon(f), \varepsilon(g) \rangle \\
&= \int_{\Gamma_{[0,t]}} d\sigma \big(1 - \delta_\emptyset(\sigma)\big) \left\langle \xi \pi_{\hat{f}}(\sigma), \widetilde{v}_{\#\sigma}(v) \eta \pi_{\hat{g}}(\sigma) \right\rangle \langle \varepsilon(f), \varepsilon(g) \rangle \\
&= \langle \xi\varepsilon(f), k_t(v) \eta\varepsilon(g) \rangle - \langle \xi, \theta(v)\eta \rangle \langle \varepsilon(f), \varepsilon(g) \rangle,
\end{aligned}$$

so k satisfies the equation weakly.

Fix $T > 0$ and $f \in \mathbb{S}$. The estimate (2.4.6) yields (for any $0 < t < T$,

$v \in \mathsf{V}$, $\xi \in \mathsf{h}$)

$$\|k_t(v)\xi\varepsilon(f)\|^2 \leq \sum_{n=0}^{\infty}(C_{f,T})^n \int_{\Delta_n[0,t]} \|\phi\|_{\mathrm{cb}}^{2n}\|\theta\|_{\mathrm{cb}}^2 M_f^{2n}\|v\|^2\|\xi\|^2\, d\boldsymbol{s},$$

where $M_f := \max\{\|\hat{d}\| : d \in \operatorname{Ran} f\}$. This implies (recall the notation introduced after (2.3.5))

$$\|k_{t,\varepsilon(f)}\| \leq C_{f,T,\|\phi\|_{\mathrm{cb}}}\|\theta\|_{\mathrm{cb}}, \tag{4.3.6}$$

for some constant $C_{f,T,\|\phi\|_{\mathrm{cb}}} > 0$. Thus $k \in \mathbb{P}_{\mathrm{sr}}(\mathsf{V};\mathsf{W},\mathcal{E}) \subset \mathbb{P}_{\mathrm{wr}}(\mathsf{V};\mathsf{W},\mathcal{E})$. This fact will be used below in conjunction with certain functorial property of the processes constructed with the help of the map $\widetilde{v}^{\theta,\phi}$ to prove the analog of (4.3.6) for the completely bounded norm.

To this end, choose $n \in \mathbb{N}$ and consider the maps

$$\theta^{(n)} : M_n(\mathsf{V}) \to M_n(\mathsf{W})$$

and

$$\widetilde{\phi^{(n)}} := \tau \circ \phi^{(n)} : M_n(\mathsf{V}) \to M_{\widehat{\mathsf{k}}}(M_n(\mathsf{V})),$$

where now τ denotes a tensor flip $B(\mathsf{h} \otimes \widehat{\mathsf{k}} \otimes \mathbb{C}^n) \to B(\mathsf{h} \otimes \mathbb{C}^n \otimes \widehat{\mathsf{k}})$. Denote $\widetilde{k} = \Lambda \circ \widetilde{v}^{\theta^{(n)},\widetilde{\phi^{(n)}}}$. Then by the first part of the proof $\widetilde{k} \in \mathbb{P}_{\mathrm{sr}}(M_n(\mathsf{V}); M_n(\mathsf{W}),\mathcal{E})$ is a weak solution of (4.3.1) with θ and ϕ replaced respectively by $\theta^{(n)}$ and $\widetilde{\phi^{(n)}}$. Moreover by (4.3.6) for $0 < t < T$

$$\|\widetilde{k}_{t,\varepsilon(f)}\| \leq C_{f,T,\|\widetilde{\phi^{(n)}}\|_{\mathrm{cb}}}\|\theta^{(n)}\|_{\mathrm{cb}} = C_{f,T,\|\phi\|_{\mathrm{cb}}}\|\theta\|_{\mathrm{cb}}. \tag{4.3.7}$$

Further consider the equality

$$\tau' \circ k^{(n)}_{t,\varepsilon(f)} = \widetilde{k}_{t,\varepsilon(f)}, \tag{4.3.8}$$

where this time τ' denotes the flip yielding the canonical complete isometry $M_n\left(C_{\mathcal{F}}(\mathsf{V})\right) \cong C_{\mathcal{F}}\left(M_n(\mathsf{V})\right)$. To prove (4.3.8), define a process $k' \in$

$\mathbb{P}_{\mathrm{sr}}(M_n(\mathsf{V}); M_n(\mathsf{W}), \mathcal{E})$ by the formula

$$k'(\widetilde{v})\widetilde{\xi}\varepsilon(f) = \left(\tau' \circ k^{(n)}_{t,\varepsilon(f)}(\widetilde{v})\right)\widetilde{\xi},$$

for all $\widetilde{v} \in M_n(\mathsf{V})$, $f \in \mathbb{S}$, $t \geq 0$, $\widetilde{\xi} \in \mathsf{h}^{\oplus n}$. As both processes $\widetilde{k}$ and k' are weakly regular, to obtain (4.3.8) it is enough to prove that k' satisfies weakly (4.3.1) with θ and ϕ replaced respectively by $\theta^{(n)}$ and $\widetilde{\phi^{(n)}}$. Choose then any $\widetilde{v} = [v_{i,j}]_{i,j=1}^n \in M_n(\mathsf{V})$, $f, g \in \mathbb{S}$, $t \geq 0$, $\widetilde{\xi} = \xi_1 \oplus \cdots \oplus \xi_n \in \mathsf{h}^{\oplus n}$, $\widetilde{\eta} = \eta_1 \oplus \cdots \oplus \eta_n \in \mathsf{h}^{\oplus n}$ and compute:

$$\left\langle \widetilde{\xi}\varepsilon(f), \left(k'_t(\widetilde{v}) - \theta^{(n)}(\widetilde{v})\right)\widetilde{\eta}\varepsilon(g)\right\rangle = \sum_{i,j=1}^n \langle \xi_i\varepsilon(f), (k_t(v_{i,j}) - \theta(v_{i,j}))\,\eta_j\varepsilon(g)\rangle$$

$$\begin{aligned}
&= \sum_{i,j=1}^n \int_0^t \left\langle \xi_i\varepsilon(f), k_s\left(E^{\hat{f}(s)}\phi(v_{i,j})E_{\hat{g}(s)}\right)\eta_j\varepsilon(g)\right\rangle ds \\
&= \sum_{i,j=1}^n \int_0^t \left\langle \xi_i\varepsilon(f), k_{s,\varepsilon(g)}\left(E^{\hat{f}(s)}\phi(v_{i,j})E_{\hat{g}(s)}\right)\eta_j\right\rangle ds \\
&= \int_0^t \left\langle \widetilde{\xi}\varepsilon(f), \tau' \circ k^{(n)}_{s,\varepsilon(g)}\left(E^{\hat{f}(s)}\widetilde{\phi^{(n)}}(\widetilde{v})E_{\hat{g}(s)}\right)\widetilde{\eta}\right\rangle ds \\
&= \int_0^t \left\langle \widetilde{\xi}\varepsilon(f), k'_s\left(E^{\hat{f}(s)}\widetilde{\phi^{(n)}}(\widetilde{v})E_{\hat{g}(s)}\right)\widetilde{\eta}\varepsilon(g)\right\rangle ds.
\end{aligned}$$

The equality (4.3.8), estimates (4.3.6) and (4.3.7) and the fact that τ' is a (complete) isometry imply the estimate

$$\|k_{t,\varepsilon(f)}\|_{\mathrm{cb}} \leq C_{f,T,\|\phi\|_{\mathrm{cb}}}\|\theta\|_{\mathrm{cb}}. \tag{4.3.9}$$

Thus k has completely bounded columns and is cb-strongly regular. The estimate (4.3.9) shows that the process K introduced in (4.3.3) is also cb-strongly regular (in fact even cb-Hölder continuous), so in particular locally square integrable. This ends the proof. □

Thus the QS differential equation (4.3.1) has a unique weak solution; it is

also a strong solution, it is cb-Hölder continuous and is given by $\Lambda \circ \widetilde{\upsilon}^{\theta,\phi}$. In the sequel we denote it by $k^{\theta,\phi}$, simplified to k^{ϕ} if $\mathsf{V} = \mathsf{W}$, $\theta = \mathrm{id}_{\mathsf{V}}$.

Remark 4.3.4. The abundance of flips in the above computations is an unavoidable consequence of putting the underlying operator space of a matrix space always 'on the left' (so for example $M_n(\mathsf{V}) = \mathsf{V} \otimes \mathbb{C}^n$) and the Fock space always 'on the right'. The advantages (certainly subjective) of such a convention were visible for example when iterated QS integrals were defined.

The reader may have noticed that throughout the proof the essential work has been done with columns of processes in question. To facilitate such operations it may be convenient to view QS processes on operator spaces as families of linear maps (indexed by time and exponential vectors) taking values in possibly unbounded column spaces. This point of view is exploited in [LS$_3$], where the existence and uniqueness of solutions are proved when the stochastic generator has completely bounded columns.

Properties of the solution

Lemma 4.3.5. *Let V, W be operator spaces, $\phi \in CB\big(\mathsf{V}; M_{\widehat{\mathsf{k}}}(\mathsf{V})\big)$, $\theta \in CB(\mathsf{V}; \mathsf{W})$. Then $k^{\theta,\phi} \in \mathbb{P}^{\dagger}_{\mathrm{cbsr}}(\mathsf{V}; \mathsf{W}, \mathcal{E})$ and the following hold.*

(a) *If $\phi' \in CB\big(\mathsf{V}; M_{\widehat{\mathsf{k}}}(\mathsf{V})\big)$, $\theta' \in CB(\mathsf{V}; \mathsf{W})$, then $k^{\theta,\phi} = k^{\theta',\phi'}$ if and only if $\theta = \theta'$ and $(\theta^{(\widehat{\mathsf{k}})} \circ \phi)^{(\widehat{\mathsf{k}}^{\otimes(n-1)})} \circ \phi_{n-1}$ for all $n \in \mathbb{N}$.*

(b) *If V and W are unital then $k^{\theta,\phi}$ is unital if and only if $\theta(1) = 1$, $\phi(1) = 0$.*

(c) *If V, W are closed under adjoint operation then $(k^{\theta,\phi})^{\dagger} = k^{\theta^{\dagger},\phi^{\dagger}}$. In particular, $k^{\theta,\phi}$ is real if and only if θ and $\theta^{(\widehat{\mathsf{k}})} \circ \phi$ are real.*

Proof. The first statement follows directly from the proof of Theorem 4.3.3 and the formula (2.4.11).
(a) Note an obvious relation $k^{\theta,\phi} - k^{\theta',\phi} = k^{\theta-\theta',\phi}$. Further $\widetilde{\upsilon}^{\theta,\phi} = \widetilde{\upsilon}^{\theta',\phi'}$ if and only if $\theta = \theta'$ and

$$\upsilon_n^{\theta,\phi} = (\theta^{(\widehat{\mathsf{k}})} \circ \phi)^{(\widehat{\mathsf{k}}^{\otimes(n-1)})} \circ \phi_{n-1}$$

for all $n \in \mathbb{N}$. The claim follows now from injectivity of the map Λ.
(b) follows again from injectivity of Λ.
(c) is an immediate consequence of (2.4.11). $\square$

Lemma 4.3.6. *Let $\mathsf{V}, \mathsf{W}_1, \mathsf{W}_2$ be operator spaces, $\phi \in CB\big(\mathsf{V}; M_{\widehat{\mathsf{k}}}(\mathsf{V})\big)$, $\theta_1 \in CB(\mathsf{V}; \mathsf{W}_1)$, $\theta_2 \in CB(\mathsf{W}_1; \mathsf{W}_2)$. Then for all $t \geq 0$, $f \in \mathbb{S}$*

$$k^{\theta_2 \circ \theta_1, \phi}_{t, \varepsilon(f)} = \theta_2^{(|\mathcal{F}\rangle)} \circ k^{\theta_1, \phi}_{t, \varepsilon(f)} \tag{4.3.10}$$

(recall that $\theta_2^{(|\mathcal{F}\rangle)}$ denotes the column lifting of the (completely bounded) map θ_2 to a map $C_{\mathcal{F}}(\mathsf{W}_1) \to C_{\mathcal{F}}(\mathsf{W}_2)$). When $k^{\theta_1, \phi}$ is completely bounded, then so is $k^{\theta_2 \circ \theta_1, \phi}$ and for all $t \geq 0$

$$k^{\theta_2 \circ \theta_1, \phi}_t = \theta_2^{(\mathcal{F})} \circ k^{\theta_1, \phi}_t.$$

Proof. For the proof of the first statement it is enough to check that the process $k' \in \mathbb{P}_{\mathrm{sr}}(\mathsf{V}; \mathsf{W}_2, \mathcal{E})$ whose columns are given by the right hand side of (4.3.10) satisfies weakly the equation (4.3.1) with $\theta := \theta_2 \circ \theta_1$, and apply Theorem 4.3.1. Let $f, g \in \mathbb{S}$, $v \in \mathsf{V}$, $t \geq 0$.

$$E^{\varepsilon(f)} \left(k'_t(v) - \theta(v) \otimes I_{\mathcal{F}}\right) E_{\varepsilon(g)} = E^{\varepsilon(f)} \left(k'_{t, \varepsilon(g)}(v) - \theta(v) \otimes E_{\varepsilon(g)}\right)$$

$$= E^{\varepsilon(f)} \left(\theta_2^{(|\mathcal{F}\rangle)} \circ k^{\theta_1, \phi}_{t, \varepsilon(g)}(v) - \theta(v) \otimes E_{\varepsilon(g)}\right) = \theta_2 \left(E^{\varepsilon(f)} \left(k^{\theta_1, \phi}_t(v) - \theta_1(v) \otimes I_{\mathcal{F}}\right) E_{\varepsilon(g)}\right)$$

$$= \theta_2 \left(\int_0^t E^{\varepsilon(f)} k^{\theta_1, \phi}_s \left(E^{\hat{f}(s)} \phi(v) E_{\hat{g}(s)}\right) E_{\varepsilon(g)} ds\right) = \int_0^t E^{\varepsilon(f)} k'_s \left(E^{\hat{f}(s)} \phi(v) E_{\hat{g}(s)}\right) E_{\varepsilon(g)},$$

where the last equality is justified due to boundedness of θ_2 and the fact that the integrands above are piecewise constant. This implies that k' is indeed a weak solution of the equation in question.

The second statement follows directly from the first. $\square$

Remark 4.3.7. Observe that if $\mathsf{V} = \mathsf{W}$ and θ 'commutes' with ϕ, i.e. $\phi \circ \theta = \theta^{(\widehat{\mathsf{k}})} \circ \phi$, then $k^\phi \circ \theta = k^{\theta, \phi}$.

Note the following fact:

Fact 4.3.8 (Continuous dependence on initial conditions). *Let* V, W *be operator spaces,* $\phi \in CB\big(\mathsf{V}; M_{\widehat{\mathsf{k}}}(\mathsf{V})\big)$*,* $\theta \in CB(\mathsf{V}; \mathsf{W})$*. Let* $(\theta_n)_{n=1}^{\infty}$ *be a sequence of maps in* $CB(\mathsf{V}; \mathsf{W})$ *convergent (in* cb*-norm) to* θ*. Then for each* $f \in \mathbb{S}$*,* $t \geq 0$ *the sequence* $(k^{\theta_n,\phi}_{t,\varepsilon(f)})_{n=1}^{\infty}$ *is convergent (in* cb*-norm) to* $k^{\theta,\phi}_{t,\varepsilon(f)}$*, and the convergence is uniform with respect to* t *on bounded subsets of* $\mathbb{R}_+$*.*

Proof. The claim follows from equality $k^{\theta_n,\phi}_{t,\varepsilon(f)} - k^{\theta,\phi}_{t,\varepsilon(f)} = k^{\theta_n-\theta,\phi}_{t,\varepsilon(f)}$ and the estimate (4.3.9). □

The result below is of the utmost importance for the next two sections. Its proof follows easily from the uniqueness properties of the solutions of our QS differential equations.

Theorem 4.3.9 ([LW$_1$]). *Let* V *be an operator space,* $\phi \in CB\big(\mathsf{V}; M_{\widehat{\mathsf{k}}}(\mathsf{V})\big)$*. The process* k^{ϕ} *is a weak standard QS cocycle on* V*, whose associated semigroups are norm continuous. The generators of associated semigroups of* k^{ϕ} *are given by* $\{E^{\hat{c}}\phi(\cdot)E_{\hat{d}} : c, d \in \mathsf{k}\}$*.*

The partial converse to this result may be found in [LS$_3$], where it is proved, using the methods of Section 3.4, that each Hölder continuous standard QS cocycle on a finite-dimensional operator space with a Hölder continuous adjoint satisfies a QS differential equation of the form (4.3.1) with the initial condition given by the identity mapping.

4.4 Coalgebraic QS differential equations on OS coalgebras

In this section the general theory of QS differential equations with nontrivial initial conditions is specified to the case of coalgebraic QS differential equations. Their solutions are shown to be weak QS convolution cocycles.

Let C be an OS coalgebra, $\varphi \in CB\big(\mathsf{C}; B(\widehat{\mathsf{k}})\big)$. A *coalgebraic QS differential equation on* C (with the coefficient φ) is the equation

$$dk_t = k_t \star_\tau \varphi \, d\Lambda_t, \quad k_0 = \iota \circ \epsilon \tag{4.4.1}$$

(τ denoting a tensor flip exchanging the order of $\widehat{\mathsf{k}}$ and $\mathcal{F}$, ι indicating an ampliation). A process $k \in \mathbb{P}_{\text{wb}}(\mathsf{C};\mathcal{E})$ is a *weak solution* of the equation (3.3.1) if for all $f, g \in \mathbb{S}$, $t \geq 0, a \in \mathsf{C}$

(4.4.2)
$$\langle \varepsilon(f), \big(k_t(a) - \epsilon(a)I_{\mathcal{F}}\big)\varepsilon(g)\rangle = \int_0^t \Big((\omega_{\varepsilon(f),\varepsilon(g)} \circ k_s) \star (\omega_{\hat{f}(s),\hat{g}(s)} \circ \varphi)\Big)(a)ds.$$

Note that weak boundedness of k is sufficient for the above formula to make sense, as bounded linear functionals are automatically completely bounded.

To define strong solutions, it is necessary to repeat from Section 4.3 the construction of the process K associated with k. Assume that a process $k \in \mathbb{P}(\mathcal{C};\mathcal{E})$ has completely bounded columns and let

$$\phi = R_{B(\widehat{\mathsf{k}})}\varphi : \mathsf{C} \to \mathsf{C} \otimes B(\widehat{\mathsf{k}}). \tag{4.4.3}$$

For each $t \geq 0$, $f \in \mathbb{S}$, define

$$K_{t,\varepsilon(f)} = \tau \circ (k_{t,\varepsilon(f)})^{(\widehat{\mathsf{k}})} \circ \phi : \mathsf{C} \to C_{\mathcal{F}}\Big(B(\widehat{\mathsf{k}})\Big) = B(\widehat{\mathsf{k}}; \widehat{\mathsf{k}} \otimes \mathcal{F}),$$

where τ denotes the map implemented by the tensor flip $\mathcal{F} \otimes \widehat{\mathsf{k}} \mapsto \widehat{\mathsf{k}} \otimes \mathcal{F}$. Let the process $K \in \mathbb{P}(\mathsf{C}; B(\widehat{\mathsf{k}}), \mathcal{E})$ be defined by

$$K_t(\zeta \otimes \varepsilon(f))(a) = K_{t,\varepsilon(f)}(a)(\zeta) \tag{4.4.4}$$

($t \geq 0, \zeta \in \widehat{\mathsf{k}}, f \in \mathbb{S}$, $a \in \mathsf{C}$).

A process $k \in \mathbb{P}(\mathsf{C};\mathcal{E})$ is a *strong solution* of the equation (4.4.1) if it is a weak solution, it has completely bounded columns and the process K introduced above is locally square integrable. In this case the First Fundamental Formula implies that for each $a \in \mathsf{C}$

$$k_t(a) = \epsilon(a)I_{\mathcal{F}} + \int_0^t K_s(a)\, d\Lambda_s.$$

From the discussion above it should be clear that in fact coalgebraic QS

differential equations are a special case of QS differential equations considered in Section 4.3. This is formalised in the next lemma.

Lemma 4.4.1. *A process $k \in \mathbb{P}_{\mathrm{wb}}(\mathsf{C}; \mathcal{E})$ is a weak (respectively, strong) solution of the equation* (4.4.1) *if and only if it is a weak (resp., strong) solution of the equation* (4.3.1) *with the initial condition $\theta := \epsilon$ and the coefficient ϕ defined by* (4.4.3).

Proof. As $\phi \in CB\left(\mathsf{C}; \mathsf{C} \otimes B(\widehat{\mathsf{k}})\right) \subset CB\left(\mathsf{C}; M_{\widehat{\mathsf{k}}}(\mathsf{C})\right)$, for the case of weak solutions it is enough to check that in the context formulated in the lemma both sides of (4.3.2) and (4.4.2) coincide. This follows from the equalities:

$$\begin{aligned}\left\langle \varepsilon(f), k_s\left(E^{\hat{f}(s)}\phi(a)E_{\hat{g}(s)}\right)\varepsilon(g)\right\rangle &= \left\langle \varepsilon(f), k_s\left(E^{\hat{f}(s)}\left((\mathrm{id}_{\mathsf{C}} \otimes \varphi)\Delta(a)\right)E_{\hat{g}(s)}\right)\varepsilon(g)\right\rangle \\ &= \left\langle \varepsilon(f), k_s\left(\left(\mathrm{id}_{\mathsf{C}} \otimes \omega_{\hat{f}(s),\hat{g}(s)} \circ \varphi\right)\Delta(a)\right)\varepsilon(g)\right\rangle \\ &= \left(\left(\omega_{\varepsilon(f),\varepsilon(g)} \circ k_s\right) \otimes \left(\omega_{\hat{f}(s),\hat{g}(s)} \circ \varphi\right)\right)(\Delta(a)) \\ &= \left(\left(\omega_{\varepsilon(f),\varepsilon(g)} \circ k_s\right) \star \left(\omega_{\hat{f}(s),\hat{g}(s)} \circ \varphi\right)\right)(a).\end{aligned}$$

The equivalence of being a strong solution is now a direct consequence of the definitions. ☐

Theorem 4.4.2. *Let C be an OS coalgebra, $\varphi \in CB\left(\mathsf{C}; B(\widehat{\mathsf{k}})\right)$. The coalgebraic QS differential equation* (4.4.1) *has a unique weak solution, denoted further by l^φ; it is also a strong solution.*

Proof. As weak solutions of the equation (4.4.1) are automatically weakly regular by (2.3.4), the claims follow from Lemma 4.4.1, Theorem 4.3.1 and Theorem 4.3.3. Note that in the notation introduced after Theorem 4.3.3 $l^\varphi = k^{\epsilon,\phi}$, where, as usual, $\phi = R_{B(\widehat{\mathsf{k}})}\varphi$. ☐

Properties of the solution

Lemma 4.4.3. *Let $\varphi \in CB\left(\mathsf{C}; B(\widehat{\mathsf{k}})\right)$. The process $l^\varphi \in \mathbb{P}^{\dagger}_{\mathrm{cbsr}}(\mathsf{C}; \mathcal{E})$ and the map $\varphi \mapsto l^\varphi$ is injective.*

Proof. Straightforward consequence of Lemma 4.3.5 and the equalities $l^\varphi = k^{\epsilon,\phi}$, $\varphi = E_{B(\widehat{\mathsf{k}})}\phi$. $\square$

Lemma 4.4.4. *Let $l = l^\varphi$, $k = k^\phi$, where $\varphi \in CB\left(\mathsf{C}; B(\widehat{\mathsf{k}})\right)$ and $\phi = R_{B(\widehat{\mathsf{k}})}\varphi$. Then for all $t \geq 0$, $f \in \mathbb{S}$*

$$k_{t,\varepsilon(f)} = R_{|\mathcal{F}\rangle} l_{t,\varepsilon(f)}.$$

Proof. Let $f, g \in \mathbb{S}$, $t \geq 0$. Note the following:

$$\begin{aligned} E^{\varepsilon(f)}\left(R_{|\mathcal{F}\rangle} l_{t,\varepsilon(g)} - \mathrm{id}_{\mathsf{C}} E_{\varepsilon(g)}\right) &= R_{B(\widehat{\mathsf{k}})}\left(E^{\varepsilon(f)}\left(l_{t,\varepsilon(g)} - \epsilon(\cdot)E_{\varepsilon(g)}\right)\right) \\ &= R_{B(\widehat{\mathsf{k}})}\left(\int_0^t \left\langle \varepsilon(f), l_s\left(E^{\hat{f}(s)}\phi(\cdot)E_{\hat{g}(s)}\right)\varepsilon(g)\right\rangle ds\right) \\ &= \int_0^t E^{\varepsilon(f)} R_{|\mathcal{F}\rangle} l_{s,\varepsilon(g)}\left(E^{\hat{f}(s)}\phi(\cdot)E_{\hat{g}(s)}\right) ds, \end{aligned}$$

where the last equality can be justified as in the proof of Lemma 4.3.6. Repeating the arguments of that proof and using Theorem 4.3.1 yields the desired formula. $\square$

The above lemma in conjunction with Fact 4.2.15 yields the following.

Corollary 4.4.5. *If either l or k above is completely bounded, so is the other one. In this case $k_t = R_{B(\mathcal{F})} l_t$ for each $t \geq 0$.*

This circle of ideas allows us to formulate the counterpart of Theorem 4.3.9 for QS convolution cocycles.

Theorem 4.4.6. *Let C be an OS coalgebra, $\varphi \in CB\left(\mathsf{C}; B(\widehat{\mathsf{k}})\right)$. The process l^φ is a weak QS convolution cocycle on C, whose associated convolution semigroups are norm continuous.*

Proof. Follows immediately from Lemma 4.4.4, Lemma 4.2.15 and Theorem 4.3.9. $\square$

The following lemma paves the way for a converse of Theorem 4.4.6, to be formulated in the next section.

Fact 4.4.7. *Let $\varphi \in CB\left(\mathsf{C}; B(\widehat{\mathsf{k}})\right)$. The generators of the convolution semigroups $\lambda^{c,d}$ associated with l^φ are equal to $\omega_{\hat{c},\hat{d}} \circ \varphi$ $(c, d \in \mathsf{k})$.*

Proof. Let $\phi = R_{B(\widehat{\mathsf{k}})}\varphi$ and note that (in the notation introduced before Lemma 4.3.2) $\upsilon_n^{\epsilon,\phi} = \varphi^{\star n}$ for all $n \in \mathbb{N}$. Therefore for all $c, d \in \mathsf{k}$, $a \in \mathsf{C}$

$$\lambda_t^{c,d}(a) = \int_{\Gamma_t} d\sigma \big\langle \pi_{\hat{c}}(\sigma), \varphi^{\star\#\sigma}(a)\pi_{\hat{d}}(\sigma)\big\rangle = \sum_{n\geq 0} \frac{t^n}{n!}(\omega_{\hat{c},\hat{d}} \circ \varphi)^{\star n}(a),$$

which concludes the proof. □

The above fact may be also proved using the analogous result for the associated semigroups of a weak standard QS cocycle k^ϕ (see Theorem 4.3.9) and applying Lemma 4.4.4.

4.5 Completely positive and contractive cocycles

In this section we analyse completely positive and contractive QS convolution cocycles on C^*-hyperbialgebras. It is shown that Markov-regularity is a sufficient (and necessary) condition for such cocycles to satisfy coalgebraic QS differential equations with a completely bounded coefficient. The precise form of the stochastic generator is given for such a case.

We begin with a straightforward consequence of Lemma 4.3.5.

Fact 4.5.1. *Let A be a C^*-hyperbialgebra, $\varphi \in CB\left(\mathsf{A}; B(\widehat{\mathsf{k}})\right)$. The QS convolution cocycle l^φ is real if and only if φ is real, and unital if and only if $\varphi(1) = 0$.*

Sufficient condition for the QS convolution cocycles on C^*-hyperbialgebras to be stochastically generated

The main result concerning CPC standard QS cocycles on C^*-algebras, including a relevant converse of Theorem 4.3.9 is the following theorem, originating

in [LiP] (for the form of the generator see also [Bel]).

Theorem 4.5.2 ([LW$_2$], [LW$_3$]). *Let $\mathsf{A} \subset B(\mathsf{h})$ be a unital (nondegenerate) C^*-algebra and $k \in \mathbb{P}(\mathsf{A}; \mathsf{A}, \mathcal{E})$. Then the following are equivalent:*

(i) *k is Markov-regular, completely positive and contractive standard QS cocycle on A;*

(ii) *$k = k^\phi$, where $\phi \in CB\big(\mathsf{A}; M_{\widehat{\mathsf{k}}}(\mathsf{A})\big)$ satisfies $\phi(1) \leq 0$ and may be decomposed as follows:*

$$\phi(a) = \Psi(a) - a \otimes \Delta^{QS} - E_{\widehat{0}} a J - J^* a E^{\widehat{0}} \tag{4.5.1}$$

($a \in \mathsf{A}$) for some maps $\Psi \in CP\big(\mathsf{A}; M_{\widehat{\mathsf{k}}}(\mathsf{A}'')$ and $J \in R_{\widehat{\mathsf{k}}}(\mathsf{A}'')$.

The convolution counterpart of Theorem 4.5.2 is:

Theorem 4.5.3. *Let A be a C^*-hyperbialgebra and $l \in \mathbb{P}(\mathsf{A}; \mathcal{E})$. Then the following are equivalent:*

(i) *l is Markov-regular, completely positive and contractive QS convolution cocycle;*

(ii) *$l = l^\varphi$ where $\varphi \in CB\big(\mathsf{A}; B(\widehat{\mathsf{k}})\big)$ satisfies $\varphi(1) \leq 0$ and may be decomposed as follows*:

$$\varphi(a) = \psi(a) - \epsilon(a) \left(\Delta^{QS} + |e_0\rangle\langle\chi| + |\chi\rangle\langle e_0|\right) \tag{4.5.2}$$

($a \in \mathsf{A}$) for some map $\psi \in CP\big(\mathsf{A}; B(\widehat{\mathsf{k}})\big)$ and vector $\chi \in \widehat{\mathsf{k}}$;

(iii) *there is a unital C^*-representation (ρ, K) of A, a contraction $D \in B(\mathsf{k}; \mathsf{K})$ and a vector $\xi \in \mathsf{K}$, such that $l = l^\varphi$ where*

$$\varphi(a) = \begin{bmatrix} \langle\xi| \\ D^* \end{bmatrix} \big(\rho(a) - \epsilon(a) I_{\mathsf{K}}\big) \begin{bmatrix} |\xi\rangle & D \end{bmatrix} + \epsilon(a)\varphi(1) \tag{4.5.3}$$

$(a \in \mathsf{A})$, and $\varphi(1)$ is nonpositive with block matrix of the form

$$\begin{bmatrix} * & * \\ * & D^*D - I_{\mathsf{k}} \end{bmatrix}.$$

Proof. Note first that none of the notions and formulas in the theorem depends on the actual faithful representation of A chosen.

(ii) $\Longrightarrow$ (i)

Define $\phi = R_{B(\widehat{\mathsf{k}})}\varphi$ and assume that A is faithfully and nondegenerately represented on some Hilbert space h. Then

$$\phi(a) = \Psi(a) - a \otimes \left(\Delta^{QS} + E_{\widehat{0}}T + T^*E^{\widehat{0}}\right),$$

where $\Psi = R_{B(\widehat{\mathsf{k}})}\psi : \mathsf{A} \to \mathsf{A} \otimes B(\widehat{\mathsf{k}})$ is completely positive by Proposition 4.1.8. Moreover $\phi(1) \leq 0$. Theorem 4.5.2 shows that k^ϕ is completely positive and contractive. Corollary 4.4.5 and again Proposition 4.1.8 yield (i).

(i) $\Longrightarrow$ (ii)

Put for all $t \geq 0$ $k_t = R_{B(\mathcal{F})}l_t$. Fact 4.2.15 and Proposition 4.1.8 imply that k is a Markov-regular CPC standard QS cocycle. By Theorem 4.5.2 k equals k^ϕ for some $\phi \in CB\left(\mathsf{A}; M_{\widehat{\mathsf{k}}}(\mathsf{A})\right)$. Define

$$\varphi = \epsilon^{(\widehat{\mathsf{k}})} \circ \phi \in CB\left(\mathsf{A}; B(\widehat{\mathsf{k}})\right),$$

and let $\{\gamma_{c,d} : c, d \in \mathsf{k}\}$ be generators of the associated convolution semigroups of the cocycle l. By Theorem 4.3.9 the generators of the associated semigroups of k^ϕ are given by $\{E^{\hat{c}}\phi(\cdot)E_{\hat{d}} : c, d \in \mathsf{k}\}$. As $l_t = E_{B(\mathcal{F})}k_t$ $(t \geq 0)$, for each $c, d \in \mathsf{k}$

$$\gamma_{c,d} = E_{B(\widehat{\mathsf{k}})}\left(E^{\hat{c}}\phi(\cdot)E_{\hat{d}}\right) = \omega_{\hat{c},\hat{d}} \circ \varphi.$$

Therefore $l = l^\varphi$, as their respective associated convolution semigroups coincide.

The condition $\phi(1) \leq 0$ is clearly equivalent to $\varphi(1) \leq 0$. It remains to prove that φ may be decomposed as in (4.5.2). To this end, assume that A is

faithfully and nondegenerately represented on h in such a way that ϵ extends to a normal state on $\mathsf{A}'' \subset B(\mathsf{h})$. This can be always achieved by considering a direct sum of a faithful representation and the GNS representation with respect to ϵ or by considering the bidual of A (in both cases ϵ is even a vector state). The continuous extension of ϵ to A'' will be denoted by the same letter. Let $\phi : \mathsf{A} \to \mathsf{A} \otimes B(\mathsf{k})$ have the form (4.5.1), with $\Psi : \mathsf{A} \to M_{\widehat{\mathsf{k}}}(\mathsf{A}'')$ being completely positive and $J \in R_{\widehat{\mathsf{k}}}(\mathsf{A}'')$. Put

$$T = \epsilon^{(\langle \widehat{\mathsf{k}}|)} \circ J \in \langle \widehat{\mathsf{k}}|, \ \ \psi = \epsilon^{(\widehat{\mathsf{k}})} \circ \Psi : \mathsf{A} \to B(\widehat{\mathsf{k}}).$$

One can show that ψ is completely positive (essentially using the same techniques as ones used to prove complete boundedness for lifted maps, see [LW$_3$]). Further

$$\epsilon^{(\widehat{\mathsf{k}})} \circ (E_{\widehat{0}} a J) = \epsilon(a) E_{\widehat{0}} T,$$

which can be checked by comparison of the respective matrix elements ($\chi, \zeta \in \widehat{\mathsf{k}}$):

$$E^{\chi} \left(\epsilon^{(\widehat{\mathsf{k}})} \circ (E_{\widehat{0}} a J) \right) E_{\zeta} = \epsilon(E^{\chi}(E_{\widehat{0}} a J E_{\zeta})) = \epsilon(a) \langle \chi, \widehat{0} \rangle \epsilon(J E_{\zeta}) =$$
$$= \epsilon(a) E^{\chi} E_{\widehat{0}} T E_{\zeta} = E^{\chi} \epsilon(a) E_{\widehat{0}} T E_{\zeta}.$$

(note that the normal extension ϵ to A'' is necessarily multiplicative). Summing up, we obtain:

(4.5.4)
$$\varphi(a) = (\epsilon \otimes 1_{\mathsf{A}}) \circ \phi(a) = \epsilon^{(\widehat{\mathsf{k}})} \circ \phi(a) = \psi(a) - \epsilon(a) \left(\Delta^{QS} + E_{\widehat{0}} T + T^* E^{\widehat{0}} \right).$$

Note that the above form of φ enforces a specific form of ϕ:

$$\phi(a) = \Psi'(a) - a \otimes \left(\Delta^{QS} + E_{\widehat{0}} T + T^* E^{\widehat{0}} \right),$$

where $\Psi' = R_{B(\widehat{\mathsf{k}})} \psi : \mathsf{A} \to \mathsf{A} \otimes B(\widehat{\mathsf{k}})$ is completely positive (possibly different than Ψ with which we started) and $T \in \langle \widehat{\mathsf{k}}|$. The above shape of ϕ corresponds to $J = I_{\mathsf{h}} \otimes T$ in (4.5.1).

(ii) $\Longrightarrow$ (iii)

Let

$$\begin{bmatrix} \langle\xi| \\ D^* \end{bmatrix} \rho(\cdot) \begin{bmatrix} |\xi\rangle & D \end{bmatrix} \tag{4.5.5}$$

be a minimal Stinespring decomposition of ψ. Thus (ρ, K) is a unital C^*-representation of A, ξ is a vector in K, D is an operator in $B(\mathsf{k}; \mathsf{K})$ (and $\rho(\mathsf{A})\big(\mathbb{C}\xi + \operatorname{Ran} D\big)$ is dense in K). Identity (4.5.3) follows, with

$$\varphi(1) = \begin{bmatrix} \|\xi\|^2 - 2\operatorname{Re}\alpha & \langle D^*\xi - c| \\ |D^*\xi - c\rangle & D^*D - I_{\mathsf{k}} \end{bmatrix},$$

where $\binom{\alpha}{c} = \chi$, so (iii) holds.

(iii) $\Longrightarrow$ (ii)

Writing

$$\begin{bmatrix} t & \langle d| \\ |d\rangle & D^*D - I_{\mathsf{k}} \end{bmatrix}$$

for the block matrix form of $\varphi(1)$, φ has the form (4.5.2) where ψ is given by (4.5.5) and

$$\chi = \begin{pmatrix} \frac{1}{2}(\|\xi\|^2 - t) \\ D^*\xi - d \end{pmatrix}$$

so (ii) holds. $\square$

Precise form of the generator

The proof of the second part of the implication (ii) $\Longrightarrow$ (i) in Theorem 4.5.3 in the last section, although short and simple, has its disadvantages. First of all it requires working in a very specific representation, secondly it indirectly, via Theorem 4.5.2 uses deep Christensen-Evans theorem on quasi-innerness of derivations on represented C^*-algebras. Moreover to prove the existence of suitable dilations of CPC Markov-regular QS convolution cocycles to *-homomorphic cocycles on C^*-bialgebras yet more explicit description of the relevant stochastic generators is needed. In this section we present an alter-

native, more elementary approach to this problem, following the ideas used in characterising the structure of generators of CPC standard QS cocycles.

Various refinements and generalisations of this characterisation were published in [LW$_2$] and [LW$_3$], but the crucial analysis has been carried out in [LiP] (see also [Bel]). Adapting arguments there requires some care, and again the R-map introduced in Section 4.1 is an indispensable tool. Straightforward attempt of investigating consequences of complete positivity of QS convolution cocycles leads to highly nontrivial considerations of a proper counterpart of the conditional CPositivity condition. To the knowledge of the author, the R-map is not very helpful in that. However, it appears that nonnegative-definite kernels with values in C^*-algebras behave well under the R-map (in a sense to be clearly visible from the proof of the next proposition).

Further in this section A denotes a fixed C^*-hyperbialgebra. For any $\tau \in B(\mathsf{A})$ define $\partial\tau : \mathsf{A} \times \mathsf{A} \to \mathsf{A}$ by

$$\partial\tau(a_1, a_2) = \tau(a_1^*)a_2 - a_1^*\tau(a_2) - \tau(a_1^*)a_2 + a_1^*\tau(1)a_2, \quad a_1, a_2 \in \mathsf{A}.$$

By analogy, for any $f \in \mathsf{A}^*$ define $\partial_\epsilon f : \mathsf{A} \times \mathsf{A} \to \mathbb{C}$ by

$$\partial_\epsilon f(a_1, a_2) = f(a_1^*a_2) - \epsilon(a_1^*)f(a_2) - f(a_1^*)\epsilon(a_2) + \epsilon(a_1^*)f(1)\epsilon(a_2), \quad a_1, a_2 \in \mathsf{A}.$$

For the notion of derivations and their basic properties that will be of use further we refer to the Appendix A.

We need to start with the finite-dimensional situation. The key fact is the following result, corresponding to Theorem 4.1 in [LiP].

Lemma 4.5.4. *Assume that* k *is finite dimensional. Let* $\varphi \in CB\big(\mathsf{A}; B(\widehat{\mathsf{k}})\big)$ *and suppose that the (weak) QS convolution cocycle* $l := l^\varphi \in \mathbb{P}(\mathsf{A}; \mathcal{E}_\mathsf{k})$ *is CPC. Then there exist a unital representation* (ρ, K) *of* A*, a* (ρ,ϵ)*-derivation* $\delta : \mathsf{A} \to B(\mathbb{C}; \mathsf{K})$*, an operator* $D \in B(\mathsf{k}; \mathsf{K})$ *and a vector* $d \in \mathsf{k}$ *such that*

$$(4.5.6) \qquad \varphi(a) = \begin{bmatrix} \lambda(a) & \epsilon(a)\langle d| + \delta^\dagger(a)D \\ \epsilon(a)|d\rangle + D^*\delta(a) & D^*\rho(a)D - \epsilon(a)I_\mathsf{k} \end{bmatrix}, \quad a \in \mathsf{A},$$

where the functional λ is real,

$$\partial_\epsilon \lambda(a_1, a_2) = \delta(a_1)^* \delta(a_2), \quad a_1, a_2 \in \mathsf{A},$$

and the following minimality condition holds:

$$\mathsf{K} = \operatorname{cl} \operatorname{Lin}\{\delta(a)1 + \rho(a)Dc : a \in \mathsf{A}, c \in \mathsf{k}\}. \tag{4.5.7}$$

If $(\mathsf{K}', \rho', \delta', D')$ *is another quadruple satisfying the above conditions (except possibly the minimality condition), then there exists a unique isometry* $V : \mathsf{K} \to \mathsf{K}'$ *such that*

$$\delta'(a) = V\delta(a), \quad \rho'(a)V = V\rho(a), \quad D' = VD, \quad a \in \mathsf{A}.$$

Proof. The proof is a modification of the argument used in the proof of Lemma 4.5 in [LiP], where k is taken to be $\mathbb{C}^d$. Write φ in block matrix form:

$$\begin{bmatrix} \lambda & \widetilde{\eta} \\ \eta & \sigma - \epsilon(\cdot) I_{\mathsf{k}} \end{bmatrix}.$$

By Fact 4.5.1 the map φ is real, in particular $\widetilde{\eta} = \eta^\dagger$. By Propositions 4.1.8 and 4.4.4, $k = R_{B(\mathcal{F}_{\mathsf{k}})} l$ is a CPC standard QS cocycle and $\phi = R_{B(\widehat{\mathsf{k}})} \varphi$ is real. The map ϕ has block matrix form

$$\begin{bmatrix} \tau & \alpha^\dagger \\ \alpha & \nu - \iota \end{bmatrix},$$

where $\tau = R\lambda$, $\alpha = R_{B(\mathbb{C};\mathsf{k})}\eta$ and $\nu = R_{B(\mathsf{k})}\sigma$. Now Lemma 4.4 in [LiP] implies that the map Ψ from $\mathsf{A} \times \mathsf{A}$ to $\mathsf{A} \otimes B(\widehat{\mathsf{k}})$, there identified with $M_{d+1}(\mathsf{A})$, defined by

$$\Psi(a_1, a_2) = \begin{bmatrix} \partial\tau(a_1, a_2) & \alpha^\dagger(a_1^* a_2) - a_1^* \alpha^\dagger(a_2) \\ \alpha(a_1^* a_2) - \alpha(a_1^*) a_2 & \nu(a_1^* a_2) \end{bmatrix}, \quad a_1, a_2 \in \mathsf{A},$$

is nonnegative-definite. Observe that if $\psi : \mathsf{A} \times \mathsf{A} \to B(\widehat{\mathsf{k}})$ is defined by the formula

$$\psi(a_1, a_2) = \begin{bmatrix} \partial_\epsilon \lambda(a_1, a_2) & \eta^\dagger(a_1^* a_2) - \epsilon(a_1^*)\eta^\dagger(a_2) \\ \eta(a_1^* a_2) - \eta(a_1^*)\epsilon(a_2) & \sigma(a_1^* a_2) \end{bmatrix}, \quad a_1, a_2 \in \mathsf{A},$$

then $\psi = (\epsilon \otimes \mathrm{id}_{B(\widehat{\mathsf{k}})}) \circ \Psi$. This in turn implies that ψ is a nonnegative-definite kernel. Indeed, for any $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathsf{A}$ and $T_1, \ldots, T_n \in B(\widehat{\mathsf{k}})$

$$\sum_{i,j=1}^{n} T_i^* \psi(a_i, a_j) T_j = \big(\epsilon \otimes \mathrm{id}_{B(\widehat{\mathsf{k}})}\big)\Big(\sum_{i,j=1}^{n} (1_{\mathsf{A}} \otimes T_i^*) \Psi(a_i, a_j)(1_{\mathsf{A}} \otimes T_j)\Big) \geq 0,$$

as $(1_{\mathsf{A}} \otimes T_i)^* = (1_{\mathsf{A}} \otimes T_i^*) \in \mathsf{A} \otimes B(\widehat{\mathsf{k}})$, ϵ is CP, and Ψ is nonnegative-definite.

Now let (K, χ) be the minimal Kolmogorov construction associated with ψ. This means that χ is a map $\mathsf{A} \to B(\widehat{\mathsf{k}}; \mathsf{K})$ satisfying

$$\chi(a_1)^* \chi(a_2) = \psi(a_1, a_2), \quad a_1, a_2 \in \mathsf{A},$$
$$\mathsf{K} = \mathrm{cl}\,\mathrm{Lin}\{\chi(a)\zeta : a \in \mathsf{A}, \zeta \in \widehat{\mathsf{k}}\}.$$

Properties of ψ imply that χ is linear and bounded. Write $\chi = [\delta\ \gamma]$, where $\delta \in B\big(\mathsf{A}; B(\mathbb{C}; \mathsf{K})\big)$ and $\gamma \in B(\mathsf{A}; B(\mathsf{k}; \mathsf{K}))$. Then, for any $a, b \in \mathsf{A}$,

$$\delta(a)^* \delta(b) = \partial_\epsilon \lambda(a, b) \text{ and } \gamma(a)^* \delta(b) = \eta(a^* b) - \eta(a^*)\epsilon(b).$$

Setting $a = b = 1$ shows that $\delta(1) = 0$. Now for $u \in \mathsf{A}$ unitary, define

$$\delta_u(a) = \delta(ua) - \delta(u)\epsilon(a),\ \gamma_u(a) = \gamma(ua) \text{ and } \chi_u = [\delta_u\ \gamma_u], \quad \text{for } a \in \mathsf{A}.$$

A straightforward computation yields

$$\chi_u(a_1)^* \chi_u(a_2) = \chi(a_1)^* \chi(a_2).$$

The uniqueness of the minimal Kolmogorov construction implies the existence

of a unique isometry $\rho(u) : \mathsf{K} \to \mathsf{K}$ given by the formula

$$\rho(u)(\delta(a)1 + \gamma(a)c) = \delta(ua)1 - \delta(u)\epsilon(a) + \gamma(ua)c, \quad a \in \mathsf{A}, c \in \mathsf{k}.$$

It follows, by standard arguments, that

$$\rho(a)(\delta(b)1 + \gamma(b)c) = \delta(ab)1 - \delta(a)\epsilon(b) + \gamma(ab)c, \quad a, b \in \mathsf{A}, c \in \mathsf{k},$$

defines a bounded operator $\rho(a)$ on K. Moreover, it is easily checked that the resulting map $\rho : \mathsf{A} \to B(\mathsf{K})$ is indeed a representation of A. It immediately follows that δ is a (ρ,ϵ)-derivation and also, by the minimality and the identity $\delta(1) = 0$, that ρ is unital. Put $D = \gamma(1) \in B(\mathsf{k}; \mathsf{K})$. Then $\gamma(a) = \rho(a)D$, and furthermore $\sigma(a) = D^*\rho(a)D$ and $\eta(a) = \epsilon(a)\eta(1) + D^*\delta(a)1$. This yields (4.5.6) with $d = \eta(1)1$.

The second part of the lemma follows once more from the uniqueness of the Kolmogorov construction. □

The step from finite-dimensional to arbitrary noise dimension space follows in exactly the same way as for standard cocycles.

Lemma 4.5.5. *Assume that k is an arbitrary Hilbert space. Let $\varphi \in CB(\mathsf{A}; B(\widehat{\mathsf{k}}))$ and suppose that the (weak) QS convolution cocycle $l^\varphi \in \mathbb{P}(\mathsf{A}; \mathcal{E}_\mathsf{k})$ is CPC. Then the conclusions of Lemma 4.5.4 hold.*

Proof. Observe first that one can obtain, as in Theorem 4.5.3,

$$\varphi(1) \leq 0 \tag{4.5.8}$$

(it can be also deduced directly from the contractivity of l^φ via the Itô Formula). Let $\{\mathsf{k}_\iota : \iota \in \mathcal{I}\}$ be an indexing of the set of all finite-dimensional subspaces of k, which is partially ordered by inclusion. As in [LW$_2$] we consider finite-dimensional cut-offs of both l^φ and φ itself. For each $\iota \in \mathcal{I}$ denote by φ_ι the map $\mathsf{A} \to B(\widehat{\mathsf{k}_\iota})$ given by the formula

$$\varphi_\iota(a) = P_\iota\varphi(a)P_\iota, \quad a \in \mathsf{A},$$

where $P_\iota \in B(\widehat{\mathsf{k}})$ is the orthogonal projection onto $\widehat{\mathsf{k}}_\iota$. Setting $l^{(\iota)} = l^{\varphi_\iota}$, $\mathcal{F}^\iota = \mathcal{F}_{\mathsf{k}_\iota}$, $\mathcal{E}^\iota = \mathcal{E}_{\mathsf{k}_\iota}$ and letting $\mathbb{E}_\iota$ denote the vacuum conditional expectation map from $B(\mathcal{F}_{\mathsf{k}})$ to $B(\mathcal{F}^\iota)$, it is easy to see that $l^{(\iota)} \in \mathbb{P}(\mathsf{A}; \mathcal{E}^\iota)$ is a CPC QS convolution cocycle and that it satisfies

$$l_t^{(\iota)}(a) = \mathbb{E}_\iota[l_t^\varphi(a)], \quad a \in \mathsf{A}, t \in \mathbb{R}_+.$$

Lemma 4.5.4 yields quadruples $(\mathsf{K}_\iota, \rho_\iota, \delta_\iota, D_\iota)$, unique up to isometric isomorphism, such that for all $a \in \mathsf{A}$

$$\varphi_\iota(a) = \begin{bmatrix} \lambda(a) & \epsilon(a)\langle d_\iota| + \delta_\iota^\dagger(a) D_\iota \\ \epsilon(a)|d_\iota\rangle + D_\iota^* \delta_\iota(a) & D_\iota^* \rho_\iota(a) D_\iota - \epsilon(a) I_\iota \end{bmatrix},$$

where I_ι denotes the identity operator on k_ι.

Exploiting the uniqueness one can construct an inductive limit K of the Hilbert spaces K_ι. Denote by U_ι the respective isometry $\mathsf{K}_\iota \to \mathsf{K}$. Then there is a unital representation ρ of A on K, a (ρ,ϵ)-derivation $\delta : \mathsf{A} \to B(\mathbb{C}; \mathsf{K})$ and, for each $c \in \mathsf{k}$ a vector $c_D \in \mathsf{K}$ such that

$$\rho(a) U_\iota = U_\iota \rho_\iota(a), \quad \delta(a) = U_\iota \delta_\iota(a) \text{ and } c_D = U_\iota D_\iota c,$$

for all $\iota \in \mathcal{I}$, $a \in \mathsf{A}$ and $c \in \mathsf{k}_\iota$. The map $c \mapsto c_D$ is linear; it remains to show that it is bounded. To this end observe that, for any $\iota \in \mathcal{I}$ such that $c \in \mathsf{k}_\iota$,

$$\begin{aligned} \left\langle \begin{pmatrix} 0 \\ c \end{pmatrix}, \varphi(1) \begin{pmatrix} 0 \\ c \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} 0 \\ c \end{pmatrix}, \varphi_\iota(1) \begin{pmatrix} 0 \\ c \end{pmatrix} \right\rangle \\ &= \left\langle c, ({D_\iota}^* D_\iota - \epsilon(1) I_\iota) c \right\rangle = \|D_\iota c\|^2 - \|c\|^2 = \|c_D\|^2 - \|c\|^2, \end{aligned}$$

and inequality (4.5.8) implies that $\|c_D\| \leq \|c\|$. The operator $D \in B(\mathsf{k}; \mathsf{K})$ given by $Dc = c_D$ completes the tuple whose existence we wished to establish. Minimality holds by the construction. □

Automatic innerness of (ρ, ϵ)-derivations leads to the following theorem.

Theorem 4.5.6. *Let $\varphi \in CB(\mathsf{A}; B(\widehat{\mathsf{k}}))$, for a C^*-hyperbialgebra A, and suppose that the weak QS convolution cocycle $l^\varphi \in \mathbb{P}(\mathsf{A}; \mathcal{E}_\mathsf{k})$ is completely positive and contractive. Then there exists a tuple $(\mathsf{K}, \rho, D, \xi, d, e, t)$ constisting of a unital representation (ρ, K) of A, a contraction $D \in B(\mathsf{k}; \mathsf{K})$, vectors $\xi \in \mathsf{K}$ and $d, e \in \mathsf{k}$, and a real number t, such that*

$$\varphi(a) = \begin{bmatrix} \lambda(a) & \epsilon(a)\langle d| + \delta^\dagger(a) D \\ \epsilon(a)|d\rangle + D^*\delta(a) & D^*\rho(a)D - \epsilon(a)I_\mathsf{k} \end{bmatrix}, \tag{4.5.9}$$

$t = \lambda(1) \leq 0$, $d = (I_\mathsf{k} - D^*D)^{1/2}e$, $\|e\|^2 \leq -t$, and, for all $a \in \mathsf{A}$,

$$\delta(a) = \big(\rho(a) - \epsilon(a)\big)|\xi\rangle, \quad \lambda(a) = \epsilon(a)(t - \|\xi\|^2) + \langle \xi, \rho(a)\xi\rangle. \tag{4.5.10}$$

Proof. Lemma 4.5.5 gives the form (4.5.9) for some ρ, K, δ and D. As all (ρ, ϵ)-derivations are inner (Corollary A.7 of the appendix), there exists $\xi \in \mathsf{K}$ such that

$$\delta(a) = \rho(a)|\xi\rangle - \epsilon(a)|\xi\rangle.$$

It remains to note that

$$\varphi(1) = \begin{bmatrix} t & \langle d| \\ |d\rangle & D^*D - I_\mathsf{k} \end{bmatrix}, \tag{4.5.11}$$

and the condition $\varphi(1) \leq 0$ implies contractivity of D, negativity of t and the existence of a vector $e \in \mathsf{k}$ satisfying all the conditions above (see the characterisation of positive matrices given in Lemma 2.1 of [GLSW]). $\square$

Corollary 4.5.7. *Let $\varphi \in CB(\mathsf{A}; B(\widehat{\mathsf{k}}))$, and let $l = l^\varphi$ be completely positive and unital. Then there exist a Hilbert space K, a unital representation $\rho : \mathsf{A} \to B(\mathsf{K})$, $\zeta \in \mathsf{K}$ and isometry $D \in B(\mathsf{k}; \mathsf{K})$ such that*

$$\varphi(a) = \begin{bmatrix} \lambda(a) & \langle D^*\delta(a)| \\ |D^*\delta(a)\rangle & D^*\rho(a)D - \epsilon(a)I_\mathsf{k} \end{bmatrix},$$

where (for all $a \in \mathsf{A}$)

$$\delta(a) = \rho(a)\zeta - \epsilon(a)\zeta,$$

$$\lambda(a) = \langle \zeta, \rho(a)\zeta \rangle - \epsilon(a)\|\zeta\|^2,$$

$$\lambda(1) = 0.$$

This is exactly the form corresponding to the purely algebraic case considered in [FrS].

Remark 4.5.8. The characterisation in Theorem 4.5.6 yields, as announced in the beginning of this section, an alternative proof of the second part of the implication (i) $\Longrightarrow$ (ii) of Theorem 4.5.3. Indeed, for $\varphi : \mathsf{A} \to B(\widehat{\mathsf{k}})$ having a form (4.5.9), define $S : \widehat{\mathsf{k}} \to \mathsf{K}$ by $S = [\zeta \quad D]$ (identifying here K with $|\mathsf{K}\rangle$). Then

$$\varphi(a) = S^* \rho(a) S + \begin{bmatrix} \lambda_0(a) & \langle \epsilon(a)\xi - D^*\zeta| \\ |\epsilon(a)\xi - D^*\zeta\rangle & -\epsilon(a) I_{\mathsf{k}} \end{bmatrix} \quad (a \in \mathsf{A}),$$

where $\lambda_0(a) = \lambda(a) - \langle \zeta, \rho(a)\zeta \rangle$. Note that as $\partial_\epsilon \lambda_0(a_1, a_2) = 0$ for any $a_1, a_2 \in \mathsf{A}$, $\lambda_0 = \lambda_0(1)\epsilon$ - one can check that $\lambda_0 - \lambda_0(1)\epsilon$ is an (ϵ,ϵ)-derivation and use Corollary A.8.

Observe that $\psi : \mathsf{A} \to B(\widehat{\mathsf{k}})$ defined by

$$\psi(a) = S^* \rho(a) S, \quad a \in \mathsf{A},$$

is evidently completely positive. Putting $T = [\frac{1}{2}\lambda_0(1) \quad \xi - D^*\zeta] \in \langle \widehat{\mathsf{k}}|$ yields the representation (4.5.2).

Remark 4.5.9. The proofs of the theorems in this section can be modified so that to obtain another proof of the whole Theorem 4.5.3, avoiding any references to Theorem 4.5.2. In particular, one may prove that if l is a Markov-regular weak QS convolution cocycle satisfying weakly a coalgebraic QS differential equation (4.4.1) for a linear map $\varphi : \mathsf{A} \to \mathcal{O}(\widehat{\mathsf{k}})$, and l is completely positive and contractive, then φ must be completely bounded (this would first

require describing what is meant by coalgebraic QS differential equations with a possibly unbounded coefficient).

4.6 *-homomorphic QS convolution cocycles on C^*-bialgebras

In this section we characterise the stochastic generators of *-homomorphic (and more generally, weakly multiplicative) QS convolution cocycles on a C^*-bialgebra A in terms of structure maps on A. Two possible definitions of quantum Lévy processes on C^*-bialgebras are also proposed and a relevant version of the Schürmann Reconstruction Theorem established.

Weakly multiplicative QS convolution cocycles

As in general the processes constructed with the help of stochastic integration do not leave the exponential domain invariant, the multiplicativity of processes on C^*-algebras in principle has to be understood weakly.

Definition 4.6.1. Let A, B be C^*-algebras, $\mathsf{B} \subset B(\mathsf{h})$. A process $k \in \mathbb{P}(\mathsf{A}; \mathsf{B}, \mathcal{E})$ is called weakly multiplicative if

$$\langle k_t(a)^* \xi\varepsilon(f), k_t(b)\eta\varepsilon(g)\rangle = \langle \xi\varepsilon(f), k_t(ab)\eta\varepsilon(g)\rangle \tag{4.6.1}$$

for all $t \geq 0$, $a, b \in \mathsf{A}$, $\xi, \eta \in \mathsf{h}$, $f, g \in \mathbb{S}$.

Compare the above definition with Definition 3.5.7 and note that when $B = \mathbb{C}$, they are equivalent. If $k \in \mathbb{P}(\mathsf{A}; \mathsf{B}, \mathcal{E})$ is weakly multiplicative and real, it must be bounded. Then it (or rather its continuous extension) is *-homomorphic, so also completely bounded.

The multiplicative properties of iterated stochastic integrals in the operator-space theoretic context are reflected by the following variant of Theorem 3.4 of [LW$_4$]. The notation used in its formulation is modelled on the one used in Section 3.5; as in fact we will use only a special case of this theorem, for

the precise interpretation we refer to the original paper of J.M. Lindsay and S.J. Wills.

Theorem 4.6.2. *Let* A, $\mathsf{B} \subset B(\mathsf{h})$ *be unital* C^**-algebras,* $\phi \in CB\big(\mathsf{A}; M_{\widehat{\mathsf{k}}}(\mathsf{A})\big)$, $\theta \in CB(\mathsf{A};\mathsf{B})$. *Then the following are equivalent:*

(a) $k^{\theta,\phi} \in \mathbb{P}(\mathsf{A};\mathsf{B},\mathcal{E})$ *is weakly multiplicative;*

(b) *for all* $n \in \mathbb{N}_0$, $a, b \in \mathsf{A}$

$$\upsilon_n^{\theta,\phi}(ab) = \sum_{\lambda\cup\mu=\{1,\dots,n\}} \upsilon_{\#\lambda}^{\theta,\phi}(a)(\lambda;n)\Delta^{QS}[\lambda\cap\mu;n]\upsilon_{\#\mu}^{\theta,\phi}(b)(\mu;n)$$

In favourable circumstances, the condition (b) of the above theorem can be simplified. The following is Corollary 4.2 (α_2) of [LW$_4$].

Theorem 4.6.3. *Let* A *be a unital* C^**-algebra,* $\phi \in CB\Big(\mathsf{A}; \mathsf{A}\otimes B(\widehat{\mathsf{k}})\Big)$. *The weak standard QS cocycle* $k^\phi \in \mathbb{P}(\mathsf{A};\mathsf{A},\mathcal{E})$ *is weakly multiplicative if and only if*

(4.6.2)
$$\phi(ab) = \phi(a)(b\otimes 1_{B(\widehat{\mathsf{k}})}) + (a\otimes 1_{B(\widehat{\mathsf{k}})})\phi(b) + \phi(a)(1_{\mathsf{A}}\otimes\Delta^{QS})\phi(b), \quad a, b \in \mathsf{A}.$$

From now on let A be a fixed C^*-bialgebra. The theorem above will allow us to characterise the stochastic generators of $*$-homomomorphic QS convolution cocycles on A. Start with the following fact:

Fact 4.6.4. *Let* $\varphi \in CB\Big(\mathsf{A}; B(\widehat{\mathsf{k}})\Big)$, $\phi = R_{B(\widehat{\mathsf{k}})}\varphi$. *Then* $l := l^\varphi$ *is weakly multiplicative if and only if* $k := k^\phi$ *is weakly multiplicative.*

Proof. As usual A is assumed to be faithfully and nondegenerately represented on some Hilbert space h.

Assume first that l is weakly multiplicative. By Lemma 4.5.1 $l^\dagger = l^{\varphi^\dagger}$, and weak multiplicativity of l implies in particular that

$$\langle \xi\varepsilon(f), (\mathrm{id}_{\mathsf{A}}\otimes l_t)(xy)\eta\varepsilon(g)\rangle = \Big\langle (\mathrm{id}_{\mathsf{A}}\otimes l_t^\dagger)(x^*)\xi\varepsilon(f), (\mathrm{id}_{\mathsf{A}}\otimes l_t)(y)\eta\varepsilon(g)\Big\rangle,$$

for all $x, y \in \mathsf{A} \odot \mathsf{A}$, $t \geq 0$, $\xi, \eta \in \mathsf{h}$, $f, g \in \mathbb{S}$. Note that the formula above may be equivalently written as:
(4.6.3)

$$\left\langle \xi\varepsilon(f), \left(\mathrm{id}_{\mathsf{A}} \otimes l_{t,\varepsilon(g)}\right)(xy)\eta \right\rangle = \left\langle \left(\mathrm{id}_{\mathsf{A}} \otimes l^{\dagger}_{t,\varepsilon(f)}\right)(x^*)\xi, \left(\mathrm{id}_{\mathsf{A}} \otimes l_{t_{\varepsilon}(g)}\right)(y)\eta \right\rangle .$$

As both sides of the above equation are clearly continuous in x and y (separately), first fixing x and varying y and then reverting this procedure one can deduce that in fact the formula is valid for all $x, y \in \mathsf{A} \otimes \mathsf{A}$. Choose then $a, b \in \mathsf{A}$ and let $x = \Delta(a)$, $y = \Delta(b)$. As Δ is multiplicative, $xy = \Delta(ab)$ and with the help of Lemma 4.4.4 equation (4.6.3) takes the form

$$\left\langle \xi\varepsilon(f), k_{t,\varepsilon(g)}(ab)\eta \right\rangle = \left\langle k^{\dagger}_{t,\varepsilon(f)}(a^*)\xi, k_{t,\varepsilon(g)}(b)\eta \right\rangle ,$$

which is exactly the statement of weak multiplicativity of k.

Assume conversely that k is weakly multiplicative. By Lemma 4.3.5 $k^{\dagger} = k^{\phi^{\dagger}}$. Choose $a, b \in \mathsf{A}$, $t \geq 0$, $f, g \in \mathbb{S}$. Then

$$\begin{aligned}
\langle l_t(a)^*\varepsilon(f), l_t(b)\varepsilon(g)\rangle &= \left(l^{\dagger}_{t,\varepsilon(f)}(a^*)\right)^* l_{t,\varepsilon(g)}(b) \\
&= \left(\left(\epsilon \otimes \mathrm{id}_{|\mathcal{F}\rangle}\right) k_{t,\varepsilon(f)}(a^*)\right)^* \left(\left(\epsilon \otimes \mathrm{id}_{|\mathcal{F}\rangle}\right) k_{t,\varepsilon(g)}(b)\right) \\
&= \epsilon\left(\left(k^{\dagger}_{t,\varepsilon(f)}(a^*)\right)^* k_{t,\varepsilon(g)}(b)\right)
\end{aligned}$$

where the second equality follows from Lemma 4.4.4 and the third uses the equality

$$(\epsilon \otimes \mathrm{id}_{\langle\mathcal{F}|})(X^*)(\epsilon \otimes \mathrm{id}_{|\mathcal{F}\rangle})(Y) = \epsilon(X^*Y),$$

valid for all $X, Y \in R_{\mathcal{F}}(\mathsf{A})$. Further weak multiplicativity of k implies that

$$\epsilon\left(\left(k^{\dagger}_{t,\varepsilon(f)}(a^*)\right)^* k_{t,\varepsilon(g)}(b)\right) = \epsilon\left(E^{\varepsilon(f)} k_{t,\varepsilon(g)}(ab)\right)$$

$$= E^{\varepsilon(f)}\left((\epsilon \otimes \mathrm{id}_{\langle\mathcal{F}|}) k_{t,\varepsilon(g)}(ab)\right) = E^{\varepsilon(f)} l_{t,\varepsilon(g)}(ab),$$

where in the last equality Lemma 4.4.4 was used again. Comparison of the above formulas yields weak multiplicativity of l. □

Proposition 4.6.5. *Let $\varphi \in CB\left(\mathsf{A}; B(\widehat{\mathsf{k}})\right)$. Then $l := l^\varphi$ is weakly multiplicative if and only if*

$$\varphi(ab) = \varphi(a)\epsilon(b) + \epsilon(a)\varphi(b) + \varphi(a)\Delta^{QS}\varphi(b), \quad a, b \in \mathsf{A}. \tag{4.6.4}$$

Proof. In view of Theorem 4.6.3 and Fact 4.6.4 it is enough to show the equivalence of the conditions (4.6.2) and (4.6.4), where again $\phi = R_{B(\widehat{\mathsf{k}})}\varphi$. If (4.6.2) holds it is enough to apply the homomorphism $\epsilon \otimes \mathrm{id}_{B(\widehat{\mathsf{k}})}$ to it to obtain (4.6.4). Conversely, if (4.6.4) holds, then for all $a_1, a_2, b_1, b_2 \in \mathsf{A}$

$$\begin{aligned}(\mathrm{id}_\mathsf{A} \otimes \varphi)\left((a_1 \otimes a_2)(b_1 \otimes b_2)\right) = {} & (\mathrm{id}_\mathsf{A} \otimes \varphi)(a_1 \otimes a_2)(\mathrm{id}_\mathsf{A} \otimes \epsilon)(b_1 \otimes b_2) \\ & + (\mathrm{id}_\mathsf{A} \otimes \epsilon)(a_1 \otimes a_2)(\mathrm{id}_\mathsf{A} \otimes \varphi)(b_1 \otimes b_2) \\ & + (\mathrm{id}_\mathsf{A} \otimes \varphi)(a_1 \otimes a_2)(1_\mathsf{A} \otimes \Delta^{QS})(\mathrm{id}_\mathsf{A} \otimes \varphi)(b_1 \otimes b_2).\end{aligned}$$

By linearity for all $x, y \in \mathsf{A} \odot \mathsf{A}$

$$\begin{aligned}(\mathrm{id}_\mathsf{A} \otimes \varphi)(xy) = {} & (\mathrm{id}_\mathsf{A} \otimes \varphi)(x)(\mathrm{id}_\mathsf{A} \otimes 1_{B(\widehat{\mathsf{k}})}\epsilon)(y) \\ & + (\mathrm{id}_\mathsf{A} \otimes 1_{B(\widehat{\mathsf{k}})}\epsilon)(x)(\mathrm{id}_\mathsf{A} \otimes \varphi)(y) \\ & + (\mathrm{id}_\mathsf{A} \otimes \varphi)(x)(1_\mathsf{A} \otimes \Delta^{QS})(\mathrm{id}_\mathsf{A} \otimes \varphi)(y),\end{aligned}$$

and again by (separate) continuity the formula remains valid for all $x, y \in \mathsf{A} \otimes \mathsf{A}$. Inserting $x = \Delta(a)$, $y = \Delta(b)$ (for some $a, b \in \mathsf{A}$) and then using multiplicativity of the coproduct and the counit property yields (4.6.2). ☐

Unital *-homomorphic QS convolution cocycles

Fact 4.5.1 and Proposition 4.6.5 yield the following characterisation of the stochastic generators of unital *-homomorphic QS convolution cocycles on C^*-bialgebras.

Theorem 4.6.6. *Let A be a C^*-bialgebra, $\varphi \in CB\left(\mathsf{A}; B(\widehat{\mathsf{k}})\right)$ and let $l = l^\varphi \in \mathbb{QSCC}(\mathsf{A}; \mathcal{E})$. Then the following are equivalent:*

(i) *l is unital and *-homomorphic;*

(ii) *φ vanishes at 1_{A} and satisfies*

$$\varphi(a^*b) = \varphi(a)^*\epsilon(b) + \overline{\epsilon(a)}\varphi(b) + \varphi(a)^*\Delta^{QS}\varphi(b); \tag{4.6.5}$$

(iii) *φ has block matrix form*

$$\begin{bmatrix} \gamma & \delta^\dagger \\ \delta & \rho - \iota \circ \epsilon \end{bmatrix} \tag{4.6.6}$$

in which ι is the ampliation $z \mapsto zI_{\mathsf{k}}$;

$$\rho : \mathsf{A} \to B(\widehat{\mathsf{k}}) \text{ is a unital }^*\text{-homomorphism;} \tag{4.6.7}$$

$\delta : \mathsf{A} \to |\mathsf{k}\rangle$ *is a (ρ, ϵ)-derivation:*

$$\delta(ab) = \delta(a)\epsilon(b) + \rho(a)\delta(b); \tag{4.6.8}$$

$\gamma : \mathsf{A} \to \mathbb{C}$ *is linear and satisfies*

$$\gamma(a^*b) = \overline{\gamma(a)}\epsilon(b) + \overline{\epsilon(a)}\gamma(b) + \delta(a)^*\delta(b). \tag{4.6.9}$$

Definition 4.6.7. Let A be a C^*-bialgebra. A linear map $\varphi : \mathsf{A} \to B(\widehat{\mathsf{k}})$ is called a structure map on A if it vanishes at 1_{A} and satisfies the formula (4.6.5) for each $a, b \in \mathsf{A}$.

Proposition 4.6.8. *Every structure map φ on a C^*-bialgebra A is inner, that is there exists a unital representation $\rho : \mathsf{A} \to B(\mathsf{k})$ and a vector $\xi \in \mathsf{k}$ such that*

$$\varphi(a) = \begin{bmatrix} \langle \xi, (\rho(a) - \epsilon(a))\xi \rangle & \langle (\rho(a^*) - \epsilon(a^*))\xi| \\ |(\rho(a) - \epsilon(a))\xi\rangle & \rho(a) - \epsilon(a)1_{\mathsf{k}} \end{bmatrix}, \quad a \in \mathsf{A}.$$

In particular, φ is completely bounded.

Proof. It is clear that φ must have a matrix form (4.6.6), where ρ is a unital representation, δ is a (ρ, ϵ)-derivation, and γ is a linear functional on A satisfying (4.6.9) for all $a, b \in \mathsf{A}$. The existence of $\xi \in \mathsf{k}$ such that

$$\delta(a) = \rho(a)\xi - \epsilon(a)\xi, \quad a \in \mathsf{A},$$

follows from Corollary A.7. Then it is easy to check that the functional $\widetilde{\gamma}$: $\mathsf{A} \to \mathbb{C}$ given by

$$\widetilde{\gamma}(a) = \langle \xi, (\rho(a) - \epsilon(a))\xi \rangle, \quad a \in \mathsf{A},$$

coincides with λ on $K := \operatorname{Ker} \epsilon$. As $\mathsf{A} = (\operatorname{Lin} 1_{\mathsf{A}}) \oplus K$ as a vector space, in fact $\gamma = \widetilde{\gamma}$. The last statement follows. □

Definition 4.6.9. Every triple (γ, δ, ρ) such that $\rho : \mathsf{A} \to B(\mathsf{k})$ is a unital representation, $\delta : \mathsf{A} \to |\mathsf{k}\rangle$ is a (ρ, ϵ)-derivation and $\gamma : \mathsf{A} \to \mathbb{C}$ is a functional satisfying (4.6.9) (for all $a, b \in \mathsf{A}$) is called a k-Schürmann triple on A.

The above proposition, Theorem 4.5.3 and Theorem 4.6.6 yield

Theorem 4.6.10. *Let A be a C^*-bialgebra. There is a one-to-one correspondence between the following objects:*

(a) *Markov-regular *-homomorphic QS convolution cocycles in $\mathbb{QSCC}(\mathsf{A}; \mathcal{E})$;*

(b) *structure maps on A;*

(c) *k-Schürmann triples on A.*

The correspondence between (a) and (b) is given by $l^\varphi \leftrightarrow \varphi$.

Remark 4.6.11. It is clear that the results concerning the perturbation of QS convolution cocycles on coalgebras, presented in Section 3.7, remain valid also for QS convolution cocycles on OS coalgebras, if only the stochastic generators of the covolution cocycles in question are completely bounded and the stochastic generators of the operator cocycles implementing the perturbation are bounded. In particular, considering the perturbation by unitary (Weyl) cocycles, one obtains again the action of the Euclidean group of k on k-Schürmann triples associated with unital *-homomorphic QS convolution cocycles on a C^*-bialgebra A.

Quantum Lévy processes on C^*-bialgebras and their reconstruction from generators

Defining quantum Lévy processes on C^*-bialgebras requires certain modifications of the original, purely algebraic, definition of L. Accardi, M. Schürmann and W. von Waldenfels ([ASW], [Sch]). The problem is how to build the convolution increments of the process given that, in general, the multiplication $\mathsf{A} \odot \mathsf{A} \to \mathsf{A}$ need not extend continuously to $\mathsf{A} \otimes \mathsf{A}$. (This is a commonly met difficulty in the theory of topological quantum groups, see [Ku2]). Below we outline two ways of overcoming this obstacle.

The simplest idea is to define a quantum Lévy process using only the concept of distributions.

Definition 4.6.12. A *weak quantum Lévy process* on a C^*-bialgebra A over a unital *-algebra-with-state $(\mathcal{B}, \omega)$ is a family $\big(j_{s,t} : \mathsf{A} \to \mathcal{B}\big)_{0 \le s \le t}$ of unital *-homomorphisms such that the functionals $\lambda_{s,t} := \omega \circ j_{s,t}$ (which are automatically bounded, so also completely bounded) satisfy the following conditions, for $0 \le r \le s \le t$:

(i) $\lambda_{r,t} = \lambda_{r,s} \star \lambda_{s,t}$;

(ii) $\lambda_{t,t} = \epsilon$;

(iii) $\lambda_{s,t} = \lambda_{0,t-s}$;

(iv)
$$\omega\left(\prod_{i=1}^{n} j_{s_i,t_i}(a_i)\right) = \prod_{i=1}^{n} \lambda_{s_i,t_i}(a_i)$$
whenever $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathsf{A}$ and the intervals $[s_1, t_1[, \ldots, [s_n, t_n[$ are disjoint;

(v) $\lambda_{0,t} \to \epsilon$ pointwise as $t \to 0$.

A weak quantum Lévy process on a C^*-bialgebra A is called Markov-regular if $\lambda_{0,t} \to \epsilon$ in norm, as $t \to 0$.

The family $\lambda := \big(\lambda_{0,t}\big)_{t\geq 0}$ is a pointwise continuous convolution semigroup of functionals on A, called the *one-dimensional distribution* of the process; if the process is Markov-regular then λ has a convolution generator which is also referred to as the *generator* of the weak quantum Lévy process. Two weak quantum Lévy processes on A, j^1 over $(\mathcal{B}^1, \omega^1)$ and j^2 over $(\mathcal{B}^2, \omega^2)$, are said to be *equivalent* if they satisfy

$$\omega^1\left(\prod_{i=1}^n j^1_{s_i,t_i}(a_i)\right) = \omega^2\left(\prod_{i=1}^n j^2_{s_i,t_i}(a_i)\right)$$

for all $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathsf{A}$ and disjoint intervals $[s_1, t_1[, \ldots, [s_n, t_n[$. Clearly two weak quantum Lévy processes are equivalent if and only if their one-dimensional distributions coincide, and if they are Markov-regular then this is equivalent to the equality of their generators.

Remark 4.6.13. Note that the above definition of a weak quantum Lévy process, in contrast to the definition of a quantum Lévy process on an algebraic *-bialgebra, does not yield a recipe for expressing the joint moments of the process increments corresponding to the overlapping time intervals, such as

$$\omega(j_{r,t}(x)j_{s,t}(y)) \text{ where } 0 \leq r, s < t.$$

To achieve the latter, one would have to formulate the weak convolution increment property (wQLPi) in greater generality and assume certain commutation relations between the increments corresponding to disjoint time intervals. For other examples of investigations of the notion of independence in noncommutative probability without imposing any particular commutation relations we refer to [HKK].

As in the algebraic case, the generator of a Markov-regular weak quantum Lévy process vanishes on 1_{A}, is real and is *conditionally positive*, that is positive on the kernel of the counit. Observe that if $l \in \mathbb{P}(\mathsf{A}; \mathcal{E})$ is a unital *-homomorphic QS convolution cocycle then, defining $\mathcal{B} := B(\mathcal{F})$, $\omega := \omega_{\varepsilon(0)}$, and $j_{s,t} := \sigma_s \circ l_{t-s}$ for all $0 \leq s \leq t$, we obtain a weak quantum Lévy process

on A, called a *Fock space quantum Lévy process*, Markov-regular if l is.

The following theorem may be proved exactly along the lines of the Schürmann Reconstruction Theorem (Theorem 3.6.2); all the necessary continuity properties follow from Theorem 4.6.8.

Theorem 4.6.14. *Let γ be a real, conditionally positive linear functional on a C^*-bialgebra A vanishing at 1_{A}. Then there is a (Markov-regular) Fock space quantum Lévy process with generator γ.*

Corollary 4.6.15. *Every Markov-regular weak quantum Lévy process is equivalent to a Fock space quantum Lévy process.*

Another notion, in a sense intermediate between weak quantum Lévy processes and Fock space quantum Lévy processes, can be formulated in terms of product systems — a similar idea is mentioned in a recent paper of M. Skeide ([Ske]). Recall that a *product system of Hilbert spaces* is a 'measurable' family of Hilbert spaces $E = \{E_t : t \geq 0\}$, together with unitaries $U_{s,t} : E_s \otimes E_t \to E_{s+t}$ $(s, t \geq 0)$ satisfying the associativity relations:

$$U_{r+s,t}(U_{r,s} \otimes I_t) = U_{r,s+t}(I_r \otimes U_{s,t}), \tag{4.6.10}$$

$(r, s, t \in \mathbb{R}_+)$ where I_s denotes the identity operator on E_s. A *unit* for the product system E is a 'measurable' family $\{u(t) : t \geq 0\}$ of vectors with $u(t) \in E_t$ and $u(s+t) = U_{s,t}\big(u(s) \otimes u(t)\big)$ for all $s, t \geq 0$ (the unit is *normalised* if, for all $t \geq 0$, $\|u(t)\| = 1$). For the precise definitions we refer to [Arv]. The unitaries $U_{s,t}$ implement the isomorphisms $\sigma_{s,t} : B(E_s \otimes E_t) \to B(E_{s+t})$.

Definition 4.6.16. A *product system quantum Lévy process* on A over a product-system-with-normalised-unit (E, u) is a family $\big(j_t : \mathsf{A} \to B(E_t)\big)_{t \geq 0}$ of unital *-homomorphisms satisfying the following conditions, for $r, s \geq 0$:

(i) $j_{s+t} = \sigma_{s,t} \circ ((j_s \otimes j_t) \circ \Delta)$,

(ii) $j_0 = \iota_0 \circ \epsilon$,

(iii) $\omega_{u(t)} \circ j_t \to \epsilon$ pointwise as $t \to 0$.

where ι_0 denotes the ampliation $\mathbb{C} \to B(E_0)$.

The Fock space $\mathcal{F}$ corresponds to a product system E by setting $E_t = \mathcal{F}_{[0,t)}$, and using the obvious unitaries whose existence is due to the exponential property (2.3.1) (in fact the product system mentioned above is the product system of the E_0-semigroup $\{\sigma_t : t \geq 0\}$ introduced in (2.3.2), see [Arv]). A normalised unit Ω is given by $\Omega(t) = \varepsilon(0) \in \mathcal{F}_{[0,t[}$, $t \geq 0$. It is therefore easy to see that every Fock space quantum Lévy process is a product system quantum Lévy process.

Proposition 4.6.17. *Each product system quantum Lévy process on* A *naturally determines a weak quantum Lévy process on* A *with the same one-dimensional distribution.*

Proof. Let j be a quantum Lévy process on A over a product-system-with-normalised-unit (E, u). We use an inductive limit construction. Define $\widetilde{\mathcal{B}} := \bigcup_{t\geq 0}(B(E_t), t)$ and introduce on $\widetilde{\mathcal{B}}$ the relation: $(T, r) \equiv (S, s)$ if there is $t \geq \max\{r, s\}$ such that $\sigma_{r,t-r}(T \otimes I_{t-r}) = \sigma_{s,t-s}(S \otimes I_{t-s})$, in other words we identify operators with common ampliations. The associativity relations (4.6.10) imply that $\equiv$ is an equivalence relation. Define $\mathcal{B} = \widetilde{\mathcal{B}}/\equiv$ and introduce the structure of a unital *-algebra on $\mathcal{B}$, consistent with the pointwise operations:

$$(T, t) + (S, t) = (T + S, t), \ \ (S, t) \cdot (T, t) = (ST, t), \ \ (T, t)^* = (T^*, t)$$

($t \geq 0, S, T \in B(E_t)$). The map $\widetilde{\omega} : \widetilde{\mathcal{B}} \to \mathbb{C}$ defined by $\widetilde{\omega}(T, t) = \omega_{u(t)}(T)$ induces a state ω on $\mathcal{B}$. For $s, t \in \mathbb{R}_+$ define

$$j_{s,t} : \mathsf{A} \to \mathcal{B} \text{ by } x \mapsto [\sigma_{s,t-s}(I_s \otimes j_{t-s}(x))]_{\equiv}.$$

It is easy to see that the family $\left(j_{s,t}\right)_{0\leq s\leq t}$ is a weak quantum Lévy process on A over $(\mathcal{B}, \omega)$. ☐

The construction in the above proof, informed by the case of QS convolution cocycles, is a special case of the familiar construction of C^*-algebraic inductive

limits. The completion of $\mathcal{A}$ with respect to the norm induced from $\widetilde{\mathcal{B}}$ is a unital C^*-algebra that may be called the *C^*-algebra of finite range operators on the product system E.*

Remark 4.6.18. A version of the reconstruction theorem also holds for unital, completely positive, QS convolution cocycles on C^*-hyperbialgebras. It is easily seen that if $l \in \mathbb{P}(\mathsf{A}; \mathcal{E})$ is a Markov-regular, unital, completely positive QS convolution cocycle on a C^*-hyperbialgebra A, then the generator of its Markov convolution semigroup is real, vanishes at 1_{A} and is conditionally positive. The GNS-type construction from the proof of Theorem 4.6.14 yields a completely bounded map $\varphi : \mathsf{A} \to B(\widehat{\mathsf{k}})$ for which the cocycle l^φ is unital and completely positive according to Proposition 4.5.1 and Theorem 4.5.3 (of course there is no reason why it should be *-homomorphic, if A is not a C^*-bialgebra). Clearly the Markov convolution semigroup of l^φ coincides with that of l.

4.7 Dilations

In this section A is a fixed C^*-bialgebra, and we are concerned with the possibility of dilating completely positive and contractive QS convolution cocycles to *-homomorphic ones, possibly by extending the dimension of the noise space. We begin with casting the characterisation of generators of *-homomorphic cocycles obtained in the previous section in the form given in Theorem 4.5.6.

Proposition 4.7.1. *Let $(\mathsf{K}, \rho, D, \xi, d, t)$ be a tuple as in Theorem 4.5.6 and let φ be the map in $CB\big(\mathsf{A}; B(\widehat{\mathsf{k}})\big)$ given by the formulas* (4.5.9) *and* (4.5.10)*. Then the (weak) QS convolution cocycle $l^\varphi \in \mathbb{P}(\mathsf{A}; \mathcal{E}_{\mathsf{k}})$ is *-homomorphic if and only if the following conditions hold*:
(i) *D is a partial isometry,*
(ii) *$Dd = 0$,*
(iii) *$DD^* \in \rho(\mathsf{A})'$,*
(iv) *$t = -\|d\|^2$,*

(v) $DD^*\delta = \delta$,
where δ *is the* (ρ, ϵ)*-derivation* $a \mapsto \big(\rho(a) - \epsilon(a)I_{\mathsf{k}}\big)|\xi\rangle$.

Proof. Fact 4.5.1 and Proposition 4.6.5 imply that l is *-homomorphic if and only if φ is real and

$$\varphi(ab) = \varphi(a)\Delta^{QS}\varphi(b) + \epsilon(a)\varphi(b) + \epsilon(b)\varphi(a), \quad a, b \in \mathsf{A}. \tag{4.7.1}$$

In the language of Theorem 4.5.6, the structure relations (4.7.1) translate into the following identities:

$$\begin{aligned}
&D^*\rho(a)DD^*\rho(b)D = D^*\rho(ab)D, \\
&D^*\delta(ab) + \epsilon(ab)|d\rangle = D^*\rho(a)D\big(D^*\delta(b) + \epsilon(b)|d\rangle\big) + D^*\delta(a)\epsilon(b) + \epsilon(a)\epsilon(b)|d\rangle, \\
&\lambda(a^*b) = \big\langle D^*\delta(a)1 + \epsilon(a)d, D^*\delta(b)1 + \epsilon(b)d\big\rangle + \lambda(a^*)\epsilon(b) + \epsilon(a^*)\lambda(b),
\end{aligned}$$

for all $a, b \in \mathsf{A}$. As in Proposition 3.3 of [GLSW], this in turn may be shown to be equivalent to the conditions (i)-(v). $\square$

Additionally, l is unital and *-homomorphic if and only if (iii), (v) are satisfied, D is an isometry, $\xi = 0$, and $t = 0$.

Stochastic dilations of CPC QS convolution cocycles

This subsection is patterned on [GLSW], with all necessary modifications. Whenever the proofs in the convolution context are straightforward adaptations of ones for standard QS cocycles, only the reference and the general ideas behind the reasoning are indicated.

Definition 4.7.2. Let k_0 be a closed subspace of a standard noise Hilbert space k. A QS convolution cocycle $j \in \mathbb{P}(\mathsf{A}; \mathcal{E}_{\mathsf{k}})$ is said to be a stochastic dilation of a QS convolution cocycle $l \in \mathbb{P}(\mathsf{A}; \mathcal{E}_{\mathsf{k}_0})$ if

$$l_t = \mathbb{E}_0 \circ j_t, \quad t \geq 0,$$

where $\mathbb{E}_0$ denotes the k_0-vacuum conditional expectation (see Section 2.3).

The following result follows in exactly the same way as its counterpart for standard cocycles ([GLSW], Lemma 1.2).

Proposition 4.7.3. *Let $\varphi \in CB(\mathsf{A}; B(\widehat{\mathsf{k}}))$ and $\psi \in CB(\mathsf{A}; B(\widehat{\mathsf{k}_0}))$, and let $j = l^\varphi \in \mathbb{P}(\mathsf{A}; \mathcal{E}_\mathsf{k})$ and $l = l^\psi \in \mathbb{P}(\mathsf{A}; \mathcal{E}_{\mathsf{k}_0})$ be the respective QS convolution cocycles. Then j is a stochastic dilation of l if and only if $\psi(\cdot) = P_0\varphi(\cdot)P_0$, where $P_0 \in B(\widehat{\mathsf{k}})$ denotes the orthogonal projection onto $\widehat{\mathsf{k}_0}$.*

Remark 4.7.4. Observe that the above characterisation excludes the possibility of obtaining the exchange free dilations, — it can be seen directly from (4.7.1) that if a Markov-regular *-homomorphic QS convolution cocycle is generated by a map having the form

$$\begin{bmatrix} * & * \\ * & 0 \end{bmatrix}$$

then it is identically 0. This uses Corollary A.8. As to the creation/annihilation free dilations they are possible only for those CPC QS convolution cocycles, whose generators have the form

$$\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}.$$

Theorem 4.7.5. *Every Markov-regular completely positive and contractive QS convolution cocycle on a C^*-bialgebra A admits a Markov-regular *-homomorphic stochastic dilation.*

Proof. Let $l \in \mathbb{P}(\mathsf{A}; \mathcal{E}_{\mathsf{k}_0})$ be a Markov-regular CPC QS convolution cocycle. Then $l = l^\varphi$ for some $\varphi \in CB\big(\mathsf{A}; B(\widehat{\mathsf{k}_0})\big)$ and we can assume that φ has matrix form (4.5.9) for a tuple $(\mathsf{K}, \rho, D, \xi, d, e, t)$ with the properties described in Theorem 4.5.6. Let $\mathsf{k}_1, \mathsf{k}_2$ be Hilbert spaces, suppose that $d_1 \in \mathsf{k}_1$, $d_2 \in \mathsf{k}_2$, $D_1 \in B(\mathsf{k}_1; \mathsf{K})$ (all as yet unspecified) and consider the map $\psi : \mathsf{A} \to B(\widehat{\mathsf{k}})$,

where $\mathsf{k} := \mathsf{k}_0 \oplus \mathsf{k}_1 \oplus \mathsf{k}_2$, given by $(a \in \mathsf{A})$

(4.7.2)
$$\psi(a) = \begin{bmatrix} \lambda(a) & \epsilon(a)\langle d| + \delta^\dagger(a)D & \epsilon(a)\langle d_1| + \delta^\dagger(a)D_1 & \epsilon(a)\langle d_2| \\ \epsilon(a)|d\rangle + D^*\delta(a) & D^*\rho(a)D - \epsilon(a)I_0 & D^*\rho(a)D_1 & 0 \\ \epsilon(a)|d_1\rangle + D_1^*\delta(a) & D_1^*\rho(a)D & D_1^*\rho(a)D_1 - \epsilon(a)I_1 & 0 \\ \epsilon(a)|d_2\rangle & 0 & 0 & -\epsilon(a)I_2 \end{bmatrix},$$

with I_i denoting I_{k_i}, $i = 0, 1, 2$. Now observe that ψ can also be written in the form

$$\psi(a) = \begin{bmatrix} \lambda(a) & \epsilon(a)\langle \widetilde{d}| + \delta^\dagger(a)\widetilde{D} \\ \epsilon(a)|\widetilde{d}\rangle + \widetilde{D}^*\delta(a) & \widetilde{D}^*\rho(a)\widetilde{D} - \epsilon(a)I_{\mathsf{k}} \end{bmatrix}, \tag{4.7.3}$$

where
$$\widetilde{d} = \begin{pmatrix} d \\ d_1 \\ d_2 \end{pmatrix} \in \mathsf{k} \text{ and } \widetilde{D} = \begin{bmatrix} D & D_1 & 0 \end{bmatrix} \in B(\mathsf{k}; \mathsf{K}).$$

As ψ is clearly completely bounded, it generates a weak QS convolution cocycle $l^\psi \in \mathbb{P}(\mathsf{A}; \mathcal{E}_{\mathsf{k}})$. It follows from Proposition 4.7.3 that l^ψ is a stochastic dilation of l^φ; it remains to show that we can choose the parameters $\mathsf{k}_1, \mathsf{k}_2$, d_1, d_2 and D_1 so that l^ψ is *-homomorphic.

To this end, it suffices to put $\mathsf{k}_1 = \mathsf{K}$, $\mathsf{k}_2 = \mathbb{C}$,

$$D_1 = \big(I_1 - DD^*\big)^{\frac{1}{2}}, \;\; d_1 = De, \;\; d_2 = \sqrt{-(t + \|e\|^2)}.$$

The above definitions make sense as $\|e\| \leq -t$ and D is a contraction. It remains then to check properties (i)-(v) of Proposition 4.7.1. First note that

$$\widetilde{D}\widetilde{D}^* = DD^* + I_1 - DD^* = I_1,$$

which implies that the conditions (i), (iii) and (v) are satisfied (one can easily

check that $\widetilde{D}^*\widetilde{D}$ is a selfadjoint projection). Further we obtain (ii):

$$\widetilde{D}\widetilde{d} = D(I_0 - D^*D)^{\frac{1}{2}}e + (I_1 - DD^*)^{\frac{1}{2}}\, De = 0.$$

Finally (iv) follows since

$$\|\widetilde{d}\|^2 = \|(I_0 - D^*D)^{1/2}e\|^2 + \|De\|^2 - \big(t + \|e\|^2\big) = -t.$$

This completes the proof. □

If l in the above theorem is unital and $\dim \mathsf{K} = \dim \operatorname{Ran}(I_{\mathsf{K}} - DD^*)$, then it is possible to obtain a unital *-homomorphic dilation $j \in \mathbb{P}(\mathsf{A}; \mathcal{E}_{\mathsf{k}})$ of l (with the noise dimension space $\mathsf{k} = \mathsf{k}_0 \oplus \mathsf{K}$).

Stinespring Theorem for QS convolution cocycles

As the previous section was a variation on the theme of [GLSW], this one addresses the convolution counterpart of the problem considered in [GLW] for standard QS cocycles. We shall show (in Theorem 4.7.8) that each Markov-regular, completely positive, contractive QS convolution cocycle has a Stinespring-like decomposition in terms of a *-homomorphic cocycle perturbed by a contractive process.

First we need some remarks on QS differential equations of the type:

$$\mathrm{d}W_t = F_t\big(I_{\widehat{\mathsf{k}}} \otimes W_t\big)\,\mathrm{d}\Lambda_t, \quad W_0 = I_{\mathcal{F}}, \tag{4.7.4}$$

where $F \in \mathbb{P}(\widehat{\mathsf{k}}; \mathcal{E})$ is a bounded process. We say that W is a weak solution of the above equation if for all $f, g \in \mathbb{S}$ and $t \geq 0$

$$\langle \varepsilon(f), (W_t - I_{\mathcal{F}})\varepsilon(g)\rangle = \int_0^t \langle \hat{f}(s) \otimes \varepsilon(f), F_s(I_{\widehat{\mathsf{k}}} \otimes W_s)(\hat{g}(s) \otimes \varepsilon(g)\rangle ds.$$

The solution of the above equation is given by the iteration procedure:

$$X_t^0 = I_{\mathcal{F}},\ X_t^1 = \int_0^t F_s(I_{\widehat{\mathsf{k}}} \otimes X_s^0)d\Lambda_s,\ \cdots, X_t^{n+1} = \int_0^t F_s(I_{\widehat{\mathsf{k}}} \otimes X_s^n)d\Lambda_s,\ \cdots$$

$$W_t\varepsilon(f) := \sum_{n=0}^{\infty} X_t^n \varepsilon(f).$$

Sufficient conditions for the above heuristics to be justified are that F is strongly measurable and has locally uniform bounds; this is also sufficient for the uniqueness of strongly regular strong solutions of the equation ([GLW], Proposition 3.1). These conditions are clearly satisfied when

$$F_s = (\mathrm{id}_{B(\widehat{\mathsf{k}})} \otimes l_s)(T), \quad s \geq 0,$$

where l is a Markov-regular, CPC QS convolution cocycle and $T \in B(\widehat{\mathsf{k}}) \otimes \mathsf{A}$.

Now let j be the *-homomorphic QS convolution cocycle l^φ ($\varphi \in CB(\mathsf{A}; B(\widehat{\mathsf{k}}))$) and let $T \in B(\widehat{\mathsf{k}}) \otimes \mathsf{A}$. Assume that $W \in \mathbb{P}(\mathcal{E})$ is a bounded solution to the equation

$$\mathrm{d}W_t = (\mathrm{id}_{B(\widehat{\mathsf{k}})} \otimes j_t)(T)\big(I_{\widehat{\mathsf{k}}} \otimes W_t\big)\mathrm{d}\Lambda_t, \quad W_0 = I_{\mathcal{F}}. \tag{4.7.5}$$

We shall identify sufficient conditions for W to be a contractive process later. The next question to be addressed is: when can we expect a process $k \in \mathbb{P}(\mathsf{A}; \mathcal{E})$ defined by

$$k_t(a) = j_t(a)W_t, \quad a \in \mathsf{A},\ t \geq 0,$$

to be a Markov-regular QS convolution cocycle?

The quantum Itô formula yields

$$\begin{aligned}\big\langle \varepsilon(f), k_t(a)\varepsilon(g)\big\rangle =& \big\langle j_t(a^*)\varepsilon(f), W_t\varepsilon(g)\big\rangle \\ =& \epsilon(a)\langle \varepsilon(f), \varepsilon(g)\rangle + \\ & \int_0^t \mathrm{ds}\Big(\big\langle \tilde{j}_s(I_{\widehat{\mathsf{k}}}\otimes a^*)(\hat{f}(s)\otimes\varepsilon(f)), \tilde{j}_s(T)\widetilde{W}_s(\hat{g}(s)\otimes\varepsilon(g))\big\rangle + \\ & \quad \big\langle \tilde{j}_s(\phi(a^*))(\hat{f}(s)\otimes\varepsilon(f)), \widetilde{W}_s(\hat{g}(s)\otimes\varepsilon(g))\big\rangle + \\ & \quad \big\langle \tilde{j}_s(\phi(a^*))(\hat{f}(s)\otimes\varepsilon(f)), (\Delta^{QS}\otimes I_{\mathcal{F}})\tilde{j}_s(T)\widetilde{W}_s(\hat{g}(s)\otimes\varepsilon(g))\big\rangle\Big)\end{aligned}$$

$(f, g \in \mathbb{S}, t \geq 0)$, where $\phi = (\varphi\otimes\mathrm{id}_{\mathsf{A}})\circ\Delta$, $\tilde{j}_s = (\mathrm{id}_{B(\widehat{\mathsf{k}})}\otimes j_s)$ and $\widetilde{W}_s = I_{\widehat{\mathsf{k}}}\otimes W_s$. Defining analogously $\tilde{k}_s = (\mathrm{id}_{B(\widehat{\mathsf{k}})}\otimes k_s)$ we see that the above equation may be written as

$$\begin{aligned}&\big\langle \varepsilon(f), k_t(a)\varepsilon(g)\big\rangle = \epsilon(a)\langle \varepsilon(f), \varepsilon(g)\rangle + \\ &\int_0^t \mathrm{ds}\left(\Big\langle \hat{f}(s)\otimes\varepsilon(f), \tilde{k}_s\left((I_{\widehat{\mathsf{k}}}\otimes a)T + \phi(a) + \phi(a)(\Delta^{QS}\otimes 1_{\mathsf{A}})T\right)(\hat{g}(s)\otimes\varepsilon(g))\Big\rangle\right).\end{aligned}$$

The process k is equal to l^ψ for some $\psi \in CB(\mathsf{A}; B(\widehat{\mathsf{k}}))$ if and only if $\widetilde{\psi} := (\psi\otimes\mathrm{id}_{\mathsf{A}})\circ\Delta$ is given by

$$a \mapsto (I_{\widehat{\mathsf{k}}}\otimes a)T + \phi(a) + \phi(a)(\Delta^{QS}\otimes 1_{\mathsf{A}})T. \tag{4.7.6}$$

Note that we need to work with the left version of the R-map introduced in Section 4.1 because of the tensor flip in the definition of the coalgebraic QS differential equation (3.3). Let $\tau = (\mathrm{id}_{B(\widehat{\mathsf{k}})}\otimes\epsilon)(T) \in B(\widehat{\mathsf{k}})$. Then (4.7.6) implies that

$$\psi(a) = \epsilon(a)\tau + \varphi(a)(1 + \Delta^{QS}\tau), \tag{4.7.7}$$

and so

$$\widetilde{\psi}(a) = \tau\otimes a + \phi(a) + \phi(a)(\Delta^{QS}\tau\otimes 1_{\mathsf{A}}). \tag{4.7.8}$$

Comparing (4.7.6) with (4.7.8) yields

$$(I_{\widehat{\mathsf{k}}} \otimes a)T + \phi(a)(\Delta^{QS} \otimes 1_{\mathsf{A}})T = \tau \otimes a + \phi(a)(\Delta^{QS}\tau \otimes 1_{\mathsf{A}}). \tag{4.7.9}$$

If $T = \tau \otimes 1_{\mathsf{A}}$ then this condition is automatically satisfied. If j is unital, then $T = \tau \otimes 1_{\mathsf{A}}$ is also necessary for (4.7.9) to hold: put $a = 1_{\mathsf{A}}$ and use $\phi(1_{\mathsf{A}}) = 0$.

Observe that when $T = \tau \otimes 1_{\mathsf{A}}$ the equation (4.7.5) takes the simple form

$$\mathrm{d}W_t = (\tau \otimes U_t W_t)\mathrm{d}\Lambda_t, \quad W_0 = I_{\mathcal{F}}, \tag{4.7.10}$$

with $U_t = j_t(1)$. In this case the condition on τ assuring contractivity of W is also particularly simple.

Theorem 4.7.6. *Let $j = l^{\varphi}$ where $\varphi \in CB(\mathsf{A}; B(\widehat{\mathsf{k}}))$ and A is a C^*-bialgebra. Suppose that j is *-homomorphic and $\tau \in B(\widehat{\mathsf{k}})$ satisfies the condition*

$$\tau + \tau^* + \tau^* \Delta^{QS} \tau \leq 0. \tag{4.7.11}$$

Then the equation (4.7.10), with $U_t := j_t(1)$, has a unique contractive strong solution $W \in \mathbb{P}(\mathcal{E})$. Moreover the process $W_t^* j_t(\cdot) W_t$ is equal to l^{θ}, where

$$\theta(a) = \epsilon(a)\left(\tau^* + \tau + \tau^*\Delta^{QS}\tau\right) + (I_{\widehat{\mathsf{k}}} + \tau^*\Delta^{QS})\varphi(a)(I_{\widehat{\mathsf{k}}} + \Delta^{QS}\tau), \quad a \in \mathsf{A}.$$

Proof. The discussion before the theorem shows that the equation (4.7.10) has a unique strongly regular strong solution $W \in \mathbb{P}(\mathcal{E})$. The Itô formula yields, for $u = \sum_{i=1}^k \lambda_i \varepsilon(f_i)$, $k \in \mathbb{N}$, $\lambda_1, \dots, \lambda_k \in \mathbb{C}$, $f_1, \dots, f_k \in \mathbb{S}$,

$$\begin{aligned} \langle W_t u, W_t u \rangle - \langle u, u \rangle = \sum_{i,j=1}^k \overline{\lambda_i}\lambda_j \int_0^t \mathrm{d}s \Big(&\big\langle \hat{f}_i(s) \otimes \varepsilon(f_i), \tau \hat{f}_j(s) \otimes U_s \varepsilon(f_j) \big\rangle \\ &+ \big\langle \tau \hat{f}_i(s) \otimes U_s \varepsilon(f_i), \hat{f}_j(s) \otimes \varepsilon(f_j) \big\rangle \\ &+ \big\langle \tau \hat{f}_i(s) \otimes U_s \varepsilon(f_i), \Delta^{QS} \tau \hat{f}_j(s) \otimes U_s \varepsilon(f_j) \big\rangle \Big). \end{aligned}$$

As $U_s = j_s(1)$ and j is *-homomorphic, each U_s is a projection. Therefore

putting

$$x(s) = \sum_{i=1}^{k} \lambda_i \hat{f}_i(s) \otimes U_s \varepsilon(f_i), \quad s \in [0, t],$$

yields

$$\langle W_t u, W_t u \rangle - \langle u, u \rangle = \int_0^t \mathrm{ds} \left\langle x(s), \left((\tau + \tau^* + \tau^* \Delta^{QS} \tau) \otimes 1_{\mathcal{F}} \right) x(s) \right\rangle \leq 0.$$

It follows that W is contractive.

The proof of the second part of the theorem is a combination of the considerations before its formulation and one more application of the Itô formula. Again let $f, g \in \mathbb{S}$, $t \geq 0$, $a \in \mathsf{A}$ and $T = \tau \otimes 1_{\mathsf{A}}$, let $\tilde{j}$, $\tilde{k}$, $\widetilde{W}$ and ψ be defined as in the discussion before the theorem and set $\widetilde{\psi} = (\psi \otimes \mathrm{id}_{\mathsf{A}}) \circ \Delta$. Then

$$\begin{aligned}
\langle \varepsilon(f), W_t^* j_t(a) W_t \varepsilon(g) \rangle = & \langle W_t \varepsilon(f), j_t(a) W_t \varepsilon(g) \rangle \\
= & \epsilon(a) \langle \varepsilon(f), \varepsilon(g) \rangle + \\
& \int_0^t \mathrm{ds} \Big(\big\langle \widetilde{W}_s(\hat{f}(s) \otimes \varepsilon(f)), \tilde{k}_s(\widetilde{\psi}(a))(\hat{g}(s) \otimes \varepsilon(g)) \big\rangle + \\
& \big\langle \tilde{j}_s(T) \widetilde{W}_s(\hat{f}(s) \otimes \varepsilon(f)), \tilde{j}_s(I_{\widehat{\mathsf{k}}} \otimes a) \widetilde{W}_s(\hat{g}(s) \otimes \varepsilon(g)) \big\rangle + \\
& \big\langle \tilde{j}_s(T) \widetilde{W}_s(\hat{f}(s) \otimes \varepsilon(f)), (\Delta^{QS} \otimes 1_{\mathcal{F}}) \tilde{k}_s(\widetilde{\psi}(a))(\hat{g}(s) \otimes \varepsilon(g)) \big\rangle \Big).
\end{aligned}$$

Finally, (4.7.7) yields

$$\begin{aligned}
\big\langle \varepsilon(f), W_t^* j_t(a) W_t \varepsilon(g) \big\rangle = & \epsilon(a) \big\langle \varepsilon(f), \varepsilon(g) \big\rangle \\
& + \int_0^t \mathrm{ds} \left\langle \hat{f}(s) \otimes \varepsilon(f), \widetilde{W}_s^* \tilde{j}_s(\widetilde{\theta}(a)) \widetilde{W}_s(\hat{g}(s) \otimes \varepsilon(g)) \right\rangle.
\end{aligned}$$

where $\widetilde{\theta} = \big(\theta \otimes \mathrm{id}_{\mathsf{A}}\big) \circ \Delta$. This completes the proof. □

For each $t \geq 0$ denote the orthogonal projection from $\mathcal{F}$ onto $\mathcal{F}_{[t,\infty[}$ by $P_{\mathsf{k},[t,\infty[}$. The following result may be proved by differentiation, as with its predecessor for standard QS cocycles, Lemma 4.2 of [GLW].

Fact 4.7.7. *Let* k *be an orthogonal direct sum of Hilbert spaces:* $\mathsf{k}_0 \oplus \mathsf{k}_1$, *let*

$\varphi \in CB\big(\mathsf{A}; B(\widehat{\mathsf{k}_0})\big)$ and $\psi \in CB\big(\mathsf{A}; B(\widehat{\mathsf{k}})\big)$, and let $k^0 = l^\varphi \in \mathbb{P}(\mathsf{A}; \mathcal{E}_{\mathsf{k}_0})$ and $k = l^\psi \in \mathbb{P}(\mathsf{A}; \mathcal{E}_{\mathsf{k}})$ be the respective weak QS convolution cocycles. Then

$$k_t(a) = k_t^0(a) \otimes P_{\mathsf{k}_1,[t,\infty[}, \quad a \in \mathsf{A}, t \geq 0,$$

if and only if

$$\psi(a) = \begin{bmatrix} \varphi(a) & 0 \\ 0 & -\epsilon(a) I_1 \end{bmatrix}, \quad a \in \mathsf{A},$$

where $I_1 = I_{\mathsf{k}_1}$.

We are ready for the main theorem of this section.

Theorem 4.7.8. *Let $k \in \mathbb{P}(\mathsf{A}; \mathcal{E}_{\mathsf{k}_0})$ be a Markov-regular CPC QS convolution cocycle on a C^*-bialgebra A. Then there exists another Hilbert space k_1, a Markov-regular, *-homomorphic QS convolution cocycle $j \in \mathbb{P}(\mathsf{A}; \mathcal{E}_{\mathsf{k}})$, where $\mathsf{k} := \mathsf{k}_0 \oplus \mathsf{k}_1$, and a contractive process $W \in \mathbb{P}(\mathcal{E}_{\mathsf{k}})$, such that*

$$\widetilde{k}_t(a) = W_t^* j_t(a) W_t, \quad t \geq 0, a \in \mathsf{A},$$

where $\widetilde{k}_t(a) := k_t(a) \otimes P_{\mathsf{k}_1,[t,\infty[}$. A process W may be chosen so that it satisfies the QS differential equation

$$\text{(4.7.12)} \qquad \mathrm{d}W_t = (l \otimes U_t W_t)\mathrm{d}\Lambda_t, \quad W_0 = I_{\mathcal{F}_{\mathsf{k}}}$$

for some $l \in B(\hat{\mathsf{k}})$ in which $U \in \mathbb{P}(\mathcal{E}_{\mathsf{k}})$ is the projection-valued process given by $U_t = j_t(1)$, $t \geq 0$.

Proof. Let $\varphi \in CB\big(\mathsf{A}; B(\widehat{\mathsf{k}})\big)$ be the stochastic generator of k (so that $k = l^\varphi$) and let $(\mathsf{K}, \rho, D, \xi, d, e, t)$ be an associated tuple, as in Theorem 4.5.6. Set $\mathsf{k}_1 = \mathsf{K}$ and define $\theta : \mathsf{A} \to B(\widehat{\mathsf{k}})$ by

$$\theta(a) = \begin{bmatrix} \lambda(a) - t\epsilon(a) & 0 & \delta^\dagger(a) \\ 0 & -\epsilon(a) I_0 & 0 \\ \delta(a) & 0 & \rho(a) - \epsilon(a) I_1 \end{bmatrix}, \quad a \in \mathsf{A},$$

where I_i denotes I_{k_i}, $i = 0, 1$ and δ is the (ρ, ϵ)-derivation: $a \mapsto \big(\rho(a) - \epsilon(a) I_{\mathsf{k}}\big)|\xi\rangle$. The map θ is completely bounded and as such generates a Markov-regular weak QS convolution cocycle $j = l^\theta \in \mathbb{P}(\mathsf{A}; \mathcal{E}_{\mathsf{k}})$. It is easily checked that θ satisfies the structure relations of Theorem 4.6.6, so j is *-homomorphic. Now choose any contraction $B \in B(\mathsf{k}_1; \mathsf{k}_0)$ and define

$$\tau = \begin{bmatrix} \frac{1}{2}t & \langle\xi| & 0 \\ 0 & -I_0 & B \\ 0 & D & -I_1 \end{bmatrix} \in B(\widehat{\mathsf{k}}).$$

Then

$$\tau^* + \tau + \tau^* \Delta^{QS} \tau = \begin{bmatrix} t & \langle\xi| & 0 \\ |\xi\rangle & D^*D - I_0 & 0 \\ 0 & 0 & B^*B - I_1 \end{bmatrix} \leq 0,$$

as B is a contraction, and $\varphi(1) \leq 0$ (see (4.5.11)).

Theorem 4.7.6 yields the existence of a contractive process $W \in \mathbb{P}(\mathcal{E}_{\mathsf{k}})$ satisfying the QS differential equation (4.7.12) and shows that the process $l \in \mathbb{P}(\mathsf{A}; \mathcal{E}_{\mathsf{k}})$ given by

$$l_t(a) = W_t^* j_t(a) W_t, \quad t \geq 0, a \in \mathsf{A},$$

is equal to l^ψ where $\psi : \mathsf{A} \to B(\hat{\mathsf{k}})$ is defined by

$$
\begin{aligned}
\psi(a) &= \epsilon(a)\left(\tau^* + \tau + \tau^*\Delta^{QS}\tau\right) + (1 + \tau^*\Delta^{QS})\theta(a)(1 + \Delta^{QS}\tau) \\
&= \epsilon(a)\begin{bmatrix} t & \langle\xi| & 0 \\ |\xi\rangle & D^*D - I_0 & 0 \\ 0 & 0 & B^*B - I_1 \end{bmatrix} \\
&\quad + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & D^* \\ 0 & B^* & 0 \end{bmatrix} \cdot \begin{bmatrix} \lambda(a) - t\epsilon(a) & 0 & \delta^\dagger(a) \\ 0 & -\epsilon(a)I_0 & 0 \\ \delta(a) & 0 & \rho(a) - \epsilon(a)I_1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & B \\ 0 & D & 0 \end{bmatrix} \\
&= \epsilon(a)\begin{bmatrix} t & \langle\xi| & 0 \\ |\xi\rangle & D^*D - I_0 & 0 \\ 0 & 0 & B^*B - I_1 \end{bmatrix} \\
&\quad + \begin{bmatrix} \lambda(a) - t\epsilon(a) & \delta^\dagger(a)D & 0 \\ D^*\delta(a) & D^*\rho(a)D - \epsilon(a)D^*D & 0 \\ 0 & 0 & -\epsilon(a)B^*B \end{bmatrix} \\
&= \begin{bmatrix} \lambda(a) & \delta^\dagger(a)D + \epsilon(a)\langle\xi| & 0 \\ D^*\delta(a) + \epsilon(a)|\xi\rangle & D^*\rho(a)D - \epsilon(a)I_0 & 0 \\ 0 & 0 & -\epsilon(a)I_1 \end{bmatrix} = \begin{bmatrix} \varphi(a) & 0 \\ 0 & -\epsilon(a)I_{\mathsf{k}} \end{bmatrix}.
\end{aligned}
$$

Application of Fact 4.7.7 completes the proof. □

4.8 Examples

In this section we present several features of *-homomorphic convolution cocycles on three types of example of C^*-bialgebras - algebras of continuous functions on compact semigroups, universal C^*-algebras of discrete groups and full compact quantum groups. We focus on connections between the results obtained in this chapter and the case of purely algebraic QS convolution cocycles analysed in Chapter 3.

Whenever $\mathcal{A}$ is a (purely algebraic) *-bialgebra, D is a dense subspace of k and $\varphi \in L(\mathcal{A}; \mathcal{O}^\dagger(\widehat{D}))$ we will follow the convention of Chapter 3 and

write l^φ for a QS convolution cocycle on $\mathcal{A}$ satisfying the coalgebraic QS differential equation (3.3.1). This hopefully will not cause any confusion with $l^\varphi \in \mathbb{QSCC}(\mathsf{A};\mathcal{E})$ being a solution of (4.4.1) for a C^*-bialgebra A and a map $\varphi \in CB\left(\mathsf{A}; B(\widehat{\mathsf{k}})\right)$; it will be always clear from the context which notion is understood.

Commutative case - continuous functions on a semigroup

Let G be a compact semigroup with identity e and $\mathsf{A} = C(G)$ denote the algebra of all continuous functions on G. The algebra A has the structure of a C^*-bialgebra with coproduct given (with the help of the natural identification $C(G) \otimes C(G) \cong C(G \times G)$) by

$$\Delta(f)(s,t) = f(st), \quad s,t \in G,\ f \in \mathsf{A}, \tag{4.8.1}$$

and counit given by

$$\epsilon(f) = f(e), \quad f \in \mathsf{A}.$$

Following the standard in quantum probability idea (going back to [AFL] and beyond), any G-valued stochastic process $\{X_t : t \in \mathbb{R}_+\}$ on the probability space $(\Omega, \mathfrak{F}, \mu)$ can be dually described by a family of unital *-homomorphisms $\{l_t : t \in \mathbb{R}_+\}$ mapping A into $L_\infty(\Omega, \mathfrak{F}, \mu)$. These homomorphisms are given by

$$l_t(f) := f \circ X_t, \ \ f \in C(G), t \geq 0,$$

and determine uniquely the original process.

Recall that a process $\{X_t : t \in \mathbb{R}_+\}$ on a semigroup with identity is called a *Lévy process* if it has identically distributed, independent increments, $\mu(\{X_0 = e\}) = 1$ and the distributions of X_t weakly converge to the Dirac measure $\delta_{\{e\}}$ (the distribution of X_0) as t tends to 0. As well known, and often pointed in the literature, not all Lévy processes have generators defined on the whole of $C(G)$. In our language, this fact corresponds to the fact that not all *-homomorphisms on A are Markov-regular. Now Markov-regularity of the process corresponds

to the norm continuity of the convolution semigroup given by

$$\lambda_t(f) = \int_\Omega f \circ X_t d\mu, \quad f \in C(G), t \geq 0$$

(note that standard weak continuity of this semigroup corresponds in the algebraic formulation to the pointwise continuity of the Markov semigroup) . This yields the following:

Proposition 4.8.1. *Let $\{X_t : t \geq 0\}$ be a Lévy processes on a normal compact semigroup G. It is equivalent (in the sense of identical finite-dimensional distributions, see Section 3.6) to a Markov-regular *-homomorphic QS convolution cocycle on A if and only if it satisfies the following condition:*

$$\mu(\{X_t = e\}) \stackrel{t\to 0^+}{\longrightarrow} 1. \tag{4.8.2}$$

Proof. It is easily seen that the condition (4.8.2) implies the existence of the bounded generator $\gamma : \mathsf{A} \to \mathbb{C}$ from which the process can be reconstructed. The other direction can be seen by considering the Markov semigroup of a given QS convolution cocycle and judiciously choosing continuous functions on G with values in $[0, 1]$ and equal to 0 outside of some neighbourhood of e (using the assumption that G is a normal topological space). ☐

Processes satisfying (4.8.2) were investigated for example in [Gre]. They are called there homogenous processes of discontinuous type and their laws are shown to be compound Poisson distributions (Theorem 2.3.5 of [Gre]). Note that the results of Appendix A on innerness of k-Schürmann triples in conjunction with the definition of quantum Poisson processes in [Fra] may be interpreted as the noncommutative counterpart of the above statement.

In general all Lévy processes on a semigroup may be equivalently realised (again in the sense of equal finite-dimensional distributions) as quantum Lévy processes on *-bialgebras ([Sch], [FSc]).

Cocommutative case - group algebras

Let Γ be a discrete group. Denote by $\mathsf{A} = C^*(\Gamma)$ the enveloping C^*-algebra of the Banach algebra $l^1(\Gamma)$ ([Ped]), called a universal (or full) C^*-algebra of Γ. By construction (as the algebra of functions on Γ with finite support is dense in A), there exists a (so-called universal) unitary representation $L : \Gamma \to \mathsf{A}$ such that $\mathcal{A} := \mathrm{Lin}\{L_g : g \in \Gamma\}$ is dense in A. Due to universality the mappings Δ and ϵ defined on the image of L by:

$$\Delta(L_g) = L_g \otimes L_g, \;\; \epsilon(L_g) = 1, \;\; g \in \Gamma$$

extend to *-homomorphisms on A. It is easy to check that A equipped with the comultiplication Δ and the counit ϵ becomes a cocommutative C^*-bialgebra.

Theorem 4.8.2. *Let $\mathsf{A} = C^*(\Gamma)$ for a discrete group Γ. Then the formula*

$$W(t, g) = l_t(L_g) \quad (g \in \Gamma, t \geq 0) \tag{4.8.3}$$

defines a bijective correspondence between unital *-homomorphic QS convolution cocycles l on A and maps $W : \mathbb{R}_+ \times \Gamma \to B(\mathcal{F})$ satisfying the following conditions:

- (i) *for each $g \in \Gamma$ the family $\{W(t, g) : t \geq 0\}$ is a QS operator cocycle;*
- (ii) *for each $t \geq 0$ the family $\{W(t, g) : g \in \Gamma\}$ is a unitary representation of Γ on $\mathcal{F}$.*

Proof. Assume that $l \in \mathbb{P}(\mathsf{A}; \mathcal{F})$ is a *-homomorphic QS convolution cocycle and define a map $W : \mathbb{R}_+ \times \Gamma \to B(\mathcal{F})$ by (4.8.3). Then, for all $t, s \geq 0$,

$g, h \in \Gamma$

$$\begin{aligned} W(t+s,g) &= l_{t+s}(L_g) = (l_t \otimes (\sigma_t \circ l_s))\Delta(L_g) = l_t(L_g) \otimes \sigma_t(l_s(L_g)) \\ &= W(t,g) \otimes \sigma_t(W(s,g)), \\ W(t,g)W(t,h) &= l_t(L_g)l_t(L_h) = l_t(L_gL_h) = l_t(L_{gh}) = W(t,gh), \\ W(t,g)^* &= l_t(L_g)^* = l_t(L_g^*) = l_t(L_{g^{-1}}) = W(t,g^{-1}), \\ W(t,e) &= l_t(L_e) = l_t(1_{\mathsf{A}}) = I_{\mathcal{F}}, \quad W(0,g) = j_0(L_g) = I_{\mathcal{F}}. \end{aligned}$$

Conversely, suppose that $W : \mathbb{R}_+ \times \Gamma \to B(\mathcal{F})$ is a map satisfying the conditions (i) and (ii). For each $t \geq 0$ define the linear map $l_t : \mathcal{A} \to B(\mathcal{F})$ by

$$l_t(L_g) = W(t,g), \quad g \in \Gamma.$$

Due to universality, the map l_t continuously extends to a *-homomorphism on the whole A. Properties of W guarantee that for $a \in \mathcal{A}$ $(t, s \geq 0)$,

$$l_0(a) = \epsilon(a)I_{\mathcal{F}}, \;\; l_{t+s}(a) = (l_t \otimes \sigma_t(l_s))\Delta(a),$$

and continuity of the extension assures that the above equations remain valid for any $a \in \mathsf{A}$. □

On the level of the stochastic generators the above correspondence takes the following form.

Proposition 4.8.3. *Let $\mathcal{A} := \mathrm{Lin}\{L_g : g \in \Gamma\}$ for a discrete group Γ. Then*

$$\psi_g = \varphi(L_g), \quad g \in \Gamma,$$

determines a bijective correspondence between maps $\varphi \in L(\mathcal{A}; B(\widehat{\mathsf{k}}))$ satisfying

(4.8.4)
$$\varphi(ab) = \varphi(a)\epsilon(b) + \epsilon(a)\varphi(b) + \varphi(a)\Delta^{QS}\varphi(b), \;\; \varphi(a)^* = \varphi(a^*), \;\; \varphi(1) = 0,$$

and maps $\psi : \Gamma \to B(\widehat{\mathsf{k}})$ satisfying

$$\psi_{gh} = \psi_g + \psi_h + \psi_g \Delta^{QS} \psi_h, \quad (\psi_g)^* = \psi_{g^{-1}}, \quad \psi_e = 0; \tag{4.8.5}$$

Proof. Elementary calculation. ☐

Remark 4.8.4. Identities (4.8.5) may be considered as a special (time-independent) case of formulae (4.2-4) in [HLP]. They are equivalent to ψ having the block matrix form

$$\psi_g = \begin{bmatrix} i\lambda_g - \frac{1}{2}\|\xi_g\|^2 & -\langle \xi_g | U_g \\ |\xi_g\rangle & U_g - I_{\mathsf{k}} \end{bmatrix}, \tag{4.8.6}$$

for a unitary representation U of Γ on $\widehat{\mathsf{k}}$ and maps $\lambda : \Gamma \to \mathbb{R}$ and $\xi : \Gamma \to \mathsf{k}$ satisfying

$$\xi_{gh} = \xi_g + U_g \xi_h \text{ and } \lambda_{gh} = \lambda_g + \lambda_h - \operatorname{Im}\langle \xi_g, U_g \xi_h \rangle.$$

Observe that according to Theorem 3.5.10 each map $\varphi \in L(\mathcal{A}; B(\widehat{\mathsf{k}}))$ satisfying (4.8.4) generates a unital, real and weakly multiplicative QS convolution cocycle l^φ on $\mathcal{A}$. Further l^φ continuously extends to a *-homomorphic QS convolution cocycle on A (see Lemma 4.8.11 below). On the other hand, given a map ψ such as in (4.8.5), for each fixed $g \in \Gamma$ the QS differential equation of the form

$$W_0(g) = I_{\mathcal{F}}, \quad dW_t(g) = \psi(g) W_t(g) d\Lambda_t$$

yields a unitary cocycle $\{W_t(g) : t \geq 0\}$ ([LW$_1$]). The map $W : \mathbb{R}_+ \times \Gamma$ given by $W(t, g) = W_t(g)$ satisfies the conditions of theorem 4.8.2. One can easily see that the correspondences described in Theorems 4.8.2 and 4.8.3 are consistent with this construction.

Proposition 4.8.5. *A unital *-homomorphic QSCC l on $\mathcal{A}$ is equal to l^φ for some $\varphi \in L(\mathcal{A}; B(\widehat{\mathsf{k}}))$ if and only if it is weakly measurable.*

Proof. One direction is trivial. For the other consider the unitary cocycles asso-

ciated with l by Theorem 4.8.2. Theorem 6.7 of [LW$_1$] implies that each of these cocycles is stochastically generated (as it is weakly measurable). Denoting respective generators by ψ_g one can see that the so obtained map $\psi : \Gamma \to B(\mathsf{k})$ satisfies the conditions (4.8.5). Theorem 4.8.3 (and discussion above) imply the desired conclusion. $\square$

If a *-homomorphic QS convolution cocycle l^φ on A is Markov-regular, the automatic innerness of its stochastic generator (Corollary A.7) implies in particular that the triple (λ, ξ, U) corresponding to φ by Theorem 4.8.3 must also be inner, in the following sense: there exists a vector $\eta \in \mathsf{k}$ such that

$$\xi(g) = U(g)\eta - \eta, \quad \lambda(g) = \operatorname{Im}\langle \eta, U_g \eta \rangle.$$

Elements of a C^*-bialgebra B are called *group-like* when they satisfy $\Delta b = b \otimes b$, as the elements L_g do. On such elements the solution $\big(k_t(b)\big)_{t \geq 0}$, of the mapping QS differential equation (4.4.1), is given by the solution of the operator QS differential equation

$$dX_t = X_t d\Lambda_L(t), \quad X_0 = I_{\mathcal{F}},$$

where $L = \varphi(b) \in B(\widehat{\mathsf{k}})$. For more on this we refer to Section 4.1 of [Sch].

Compact quantum groups

The concept of compact quantum groups was introduced by S.L. Woronowicz, in [Wor$_1$]. For our purposes it is most convenient to adopt the following definition:

Definition 4.8.6 ([Wor$_2$]). A compact quantum group is a pair (A, Δ), where A is a C^*-algebra, and $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ is a unital, *-homomorphic map which is coassociative and satisfies the quantum cancellation properties:

$$\overline{\operatorname{Lin}}((1 \otimes \mathsf{A})\Delta(\mathsf{A})) = \overline{\operatorname{Lin}}((\mathsf{A} \otimes 1)\Delta(\mathsf{A})) = \mathsf{A} \otimes \mathsf{A}.$$

For the concept of Hopf *-algebras and their unitary corepresentations, as well as unitary corepresentations of compact quantum groups, we refer to Appendix B (see also [KlS]). Here it is sufficient to note the facts contained in the following theorem.

Theorem 4.8.7 ([Wor$_1$]). *Let A be a compact quantum group and let $\mathcal{A}$ denote the linear span of the matrix coefficients of irreducible unitary corepresentations of A. Then $\mathcal{A}$ is a dense *-subalgebra of A, the coproduct of A restricts to an algebraic coproduct Δ_0 on $\mathcal{A}$ and there is a natural counit ϵ and coinverse $\mathcal{S}$ on $\mathcal{A}$ which makes it a Hopf *-algebra.*

Remark 4.8.8 ([BMT]). In the above theorem $(\mathcal{A}, \Delta_0, \epsilon, \mathcal{S})$ is the unique dense Hopf *-subalgebra of A, in the following sense: if $(\mathcal{A}', \Delta_0', \epsilon', \mathcal{S}')$ is a Hopf *-algebra, in which $\mathcal{A}'$ is a dense *-subalgebra of A and the coproduct of A restricts to an algebraic coproduct Δ_0' on $\mathcal{A}'$, then $(\mathcal{A}', \Delta_0', \epsilon', \mathcal{S}')$ equals $(\mathcal{A}, \Delta_0, \epsilon, \mathcal{S})$.

The Hopf *-algebra arising here is called the *associated Hopf *-algebra* of (A, Δ). When $\mathsf{A} = C(G)$ for a compact group G, $\mathcal{A}$ is the algebra of all matrix coefficients of unitary representations of G; when A is a universal C^*-algebra of a discrete group Γ, $\mathcal{A} = \mathrm{Lin}\{L_g : g \in \Gamma\}$ (see the beginning of the previous subsection). M. Dijkhuizen and T. Koornwinder observed that the Hopf *-algebras arising in this way have intrinsic algebraic structure.

Definition 4.8.9. A Hopf *-algebra $\mathcal{A}$ is called a CQG (compact quantum group) algebra if it is the linear span of all matrix elements of its finite dimensional unitary corepresentations.

Theorem 4.8.10 ([DiK]). *Each Hopf *-algebra associated with a compact quantum group is a CQG algebra. Conversely, if $\mathcal{A}$ is a CQG algebra then*

(4.8.7)
$$\|a\| := \sup\left\{\|\pi(a)\| : \pi \textit{ is a }^*\textit{-representation of } \mathcal{A} \textit{ on a Hilbert space}\right\}$$

defines a C^*-norm on $\mathcal{A}$ and the completion of $\mathcal{A}$ with respect to this norm is a compact quantum group whose comultiplication extends that of $\mathcal{A}$.

The compact quantum group obtained in this theorem is called the *universal compact quantum group of* $\mathcal{A}$ and is denoted $\mathcal{A}_{\mathrm{u}}$.

For further use note the following extension of Lemma 11.31 in [KlS]:

Lemma 4.8.11. *Let E be a dense subset of a Hilbert space H and let $\mathcal{A}$ be a CQG algebra. Suppose that $\pi : \mathcal{A} \to \mathcal{O}^\dagger(\widehat{E})$ is a real, unital and weakly multiplicative (where the latter, as usual, means multiplicative with respect to the product $\cdot$ introduced in* (3.5.1)*). Then π admits a continuous extension to a unital *-homomorphism from $\mathcal{A}_u$ to $B(H)$.*

Proof. Let $(v_{i,j})_{i,j=1}^n$ be any finite dimensional unitary corepresentation of $\mathcal{A}$. For any $i, j \in \{1, \ldots, n\}, \xi \in E$,

$$\|\pi(v_{i,j})\xi\|^2 \leq \sum_{k=1}^n \|\pi(v_{k,j})\xi\|^2 = \sum_{k=1}^n \langle \pi(v_{k,j})\xi, \pi(v_{k,j})\xi\rangle$$

$$= \sum_{k=1}^n \langle \xi, \pi(v_{k,j})^\dagger \pi(v_{k,j})\xi\rangle = \left\langle \xi, \pi\left(\sum_{k=1}^n v_{k,j}^* v_{k,j}\right)\xi \right\rangle = \|\xi\|^2,$$

as $\sum_{k=1}^n v_{k,j}^* v_{k,j} = 1_{\mathcal{A}}$. This implies that for each $a \in \mathcal{A}$ the operator $\pi(a)$ extends to a bounded map on H. A moment of reflection suffices to see that so extended π is a unital *-homomorphism on $\mathcal{A}$. The definition of the canonical norm on $\mathcal{A}$ implies that π is contractive on $\mathcal{A}$ with respect to this norm and as such may be continuously extended to a unital representation of $\mathcal{A}_u$. ☐

Definition 4.8.12. A compact quantum group (A, Δ) is called full if the C^*-norm it induces on its associated CQG algebra $\mathcal{A}$ coincides with its canonical norm defined in (4.8.7) (equivalently, if A is *-isomorphic to $\mathcal{A}_{\mathrm{u}}$).

The notion of full compact quantum groups was introduced in [BMT] and in [BaS] (in the first paper they were called universal compact quantum groups). It is very relevant for our context, as the above facts imply the following

Proposition 4.8.13. *Each full compact quantum group A is a C^*-bialgebra (with the counit being a continuous extension of the counit of its associated Hopf *-algebra $\mathcal{A}$). There is a bijective correspondence between unital*

*-homomorphic QS convolution cocycles on A and unital, real and weakly multiplicative QS convolution cocycles on $\mathcal{A}$.

Both families of examples described in previous two subsections - algebras of continuous functions on compact groups and full C^*-algebras of discrete groups - are full compact quantum groups. Moreover most of the genuinely quantum (i.e. neither commutative nor cocommutative) compact quantum groups considered in the literature also fall into this category - among them the queen of all examples, quantum $SU_q(2)$.

Reconnecting with the contents of Chapter 3, we obtain

Theorem 4.8.14. *Let $k \in \mathbb{P}_{\mathrm{cb}}(\mathsf{A}; \mathcal{E})$, where A is a full compact quantum group with the associated Hopf *-algebra $\mathcal{A}$. Then the following are equivalent:*

(i) *k and $k^\dagger$ are Hölder continuous QS convolution cocycles;*

(ii) *$k|_{\mathcal{A}} = l^\varphi$ for some map $\varphi \in L(\mathcal{A}; B(\widehat{\mathsf{k}}))$.*

Proof. One direction follows from the fact that $\mathcal{A}$ is an (algebraic) coalgebra and Theorem 3.4.7. The other is trivial. □

Specialising to *-homomorphic cocycles yields a much stronger result:

Theorem 4.8.15. *Let $k \in \mathbb{P}(\mathsf{A}; \mathcal{E}_D)$, where A is a full compact quantum group with the associated Hopf *-algebra $\mathcal{A}$ and D is a dense subspace of k. Then the following are equivalent:*

(i) *k is Hölder continuous, unital and *-homomorphic QS convolution cocycle;*

(ii) *k is bounded and $k|_{\mathcal{A}} = l^\varphi$ for some $\varphi \in L(\mathcal{A}; \mathcal{O}^\dagger(\widehat{D}))$ satisfying the structure relations (3.5.6) and vanishing at $1_{\mathcal{A}}$.*

Proof. (i)$\Rightarrow$(ii)
Follows from the previous theorem, Theorem 3.4.7, and the implication (i)$\Rightarrow$(ii) of Theorem 3.5.10. Note that Theorem 3.4.7 yields even $\varphi \in L\left(\mathcal{A}; \mathcal{O}^\dagger(\widehat{\mathsf{k}})\right) = L\left(\mathcal{A}; B(\widehat{\mathsf{k}})\right)$.

(ii)$\Rightarrow$(i)
Theorem 3.5.10 guarantees that $l = k|_{\mathcal{A}}$ is real, unital, and weakly multiplicative. Lemma 4.8.11 shows that l admits a continuous extension to a *-homomorphic unital process in $\mathbb{P}(\mathcal{A}; \mathcal{E}_D)$ which must coincide with k. Application of the previous theorem ends the proof. $\square$

The above theorem could be equivalently formulated in the following way:

Theorem 4.8.16. *Let $k \in \mathbb{P}(\mathcal{A}; \mathcal{E}_D)$, where A is a full compact quantum group with associated Hopf *-algebra $\mathcal{A}$ and D is a dense subspace of k. Then the following are equivalent:*

(i) *k extends to a Hölder continuous unital and *-homomorphic QS convolution cocycle on A;*

(ii) *$k = l^\varphi$ for some $\varphi \in L(\mathcal{A}; \mathcal{O}^\dagger(\widehat{D}))$ satisfying the structure relations* (3.5.6) *and vanishing at $1_{\mathcal{A}}$.*

Remark 4.8.17. In the course of the proof of the last theorem it was established that each map φ defined on a CQG algebra $\mathcal{A}$ with values in $\mathcal{O}^\dagger(\widehat{D})$ satisfying the conditions (3.5.6) and vanishing at $1_{\mathcal{A}}$ must be bounded-operator-valued. However, φ need not extend to A (see examples in [ScS]). If it is continuous, then it is necessarily completely bounded.

Construction of C^*-hyperbialgebras by conditional expectations and QS convolution cocycles

In previous subsections the variety of examples of C^*-bialgebras were presented and *-homomorphic QS convolution cocycles on these were described and given alternative interpretation. Here the way of obtaining new C^*-hyperbialgebras as certain subalgebras of other C^*-hyperbialgebras is recalled. The construction was explicitly described (in the context of compact quantum groups) in the papers [Kal] and [KaCh], but its origins go back much further (see [ChV] and references therein). In fact all known examples of noncommutative C^*-hyperbialgebras arise in this way from C^*-bialgebras.

Fact 4.8.18. *Let $(\mathsf{A}, \Delta, \epsilon)$ be a C^*-hyperbialgebra. Assume that $\widetilde{\mathsf{A}}$ is a unital C^*-subalgebra of A and that there exists a conditional expectation (that is norm-one projection) P from A onto $\widetilde{\mathsf{A}}$ satisfying the following identities:*

$$(P \otimes id_{\mathsf{A}})\Delta P = (P \otimes P)\Delta = (id_{\mathsf{A}} \otimes P)\Delta.$$

Then $\widetilde{\mathsf{A}}$ equipped with the coproduct $\widetilde{\Delta}$ and the counit $\widetilde{\epsilon}$, where

$$\widetilde{\Delta} = (P \otimes P)\Delta|_{\widetilde{\mathsf{A}}}, \quad \widetilde{\epsilon} = \epsilon|_{\widetilde{\mathsf{A}}},$$

is a C^*-hyperbialgebra.

Two particular cases of this construction are *double coset bialgebras* and *Delsarte C^*-hyperbialgebras*; they are described below.

Let $(\mathsf{A}_1, \Delta_1, \epsilon_1)$ and $(\mathsf{A}_2, \Delta_2, \epsilon_2)$ be C^*-bialgebras and assume that the latter is a *quantum subsemigroup* of the former. This means that there exists a unital *-homomorphism $\pi : \mathsf{A}_1 \to \mathsf{A}_2$ which is surjective and intertwines the coalgebraic structure:

$$(\pi \otimes \pi) \circ \Delta_1 = \Delta_2 \circ \pi, \quad \epsilon_2 \circ \pi = \epsilon_1.$$

Assume additionally that A_2 admits a Haar state h (recall formula (3.1.9)). Define the following C^*-subalgebras of A_1 (*the algebras of left and right cosets of* A_2):

$$\mathsf{A}_1/\mathsf{A}_2 = \{a \in \mathsf{A}_1 : (\mathrm{id}_{\mathsf{A}_1} \otimes \pi) \circ \Delta_1(a) = a \otimes 1\},$$

$$\mathsf{A}_1\backslash\mathsf{A}_2 = \{a \in \mathsf{A}_1 : (\pi \otimes \mathrm{id}_{\mathsf{A}_1}) \circ \Delta_1(a) = 1 \otimes a\}$$

and the *double coset bialgebra*

$$\mathsf{A}_2\backslash\mathsf{A}_1/\mathsf{A}_2 = \mathsf{A}_1/\mathsf{A}_2 \cap \mathsf{A}_1\backslash\mathsf{A}_2.$$

It can be checked that the map $P : \mathsf{A}_1 \to \widetilde{\mathsf{A}} := \mathsf{A}_2\backslash\mathsf{A}_1/\mathsf{A}_2$ defined by

$$P(a) = ((h \circ \pi) \otimes \mathrm{id}_{\mathsf{A}_1} \otimes (h \circ \pi)) (\Delta_1 \otimes \mathrm{id}_{\mathsf{A}_1}) \Delta_1(a), \quad a \in \mathsf{A}_1,$$

satisfies the conditions given in Fact 4.8.18. Its action may be understood as averaging (twice) over the quantum subsemigroup; the construction is common in theory of classical hypergroups ([BlH]).

Let $(\mathsf{A}, \Delta, \epsilon)$ be a C^*-bialgebra and assume that a compact group Γ acts (continuously with respect to the topology of pointwise convergence) on A by C^*-algebra automorphisms satisfying

$$(\gamma \otimes \gamma) \circ \Delta = \Delta \circ \gamma, \ \ \epsilon \circ \gamma = \epsilon, \quad \gamma \in \Gamma.$$

Let $\widetilde{\mathsf{A}}$ be the fixed point subalgebra, $\widetilde{\mathsf{A}} = \{a \in \mathsf{A} : \forall_{\gamma \in \Gamma}\ \gamma(a) = a\}$. It is easily checked that the map $P : \mathsf{A} \to \widetilde{\mathsf{A}}$ given by

$$P(a) = \int_\Gamma \gamma(a) d\gamma, \quad a \in \mathsf{A}$$

(where $d\gamma$ denotes the normalised Haar measure on Γ), satisfies the assumptions of Fact 4.8.18. The resulting C^*-hyperbialgebra is called a *Delsarte C^*-hyperbialgebra*.

The connection between QS convolution cocycles on A and $\widetilde{\mathsf{A}}$ is given in the following fact:

Fact 4.8.19. *Assume that $(\widetilde{\mathsf{A}}, \widetilde{\Delta}, \widetilde{\epsilon})$ is a C^*-hyperbialgebra arising from a C^*-hyperbialgebra $(\mathsf{A}, \Delta, \epsilon)$ via the construction presented in Fact 4.8.18, with the associated conditional expectation P. Then there is a bijective correspondence between QS convolution cocycles on $\widetilde{\mathsf{A}}$ and P-invariant convolution increment processes on A with initial condition given by the idempotent functional $\epsilon \circ P$.*

Proof. Assume first that $\widetilde{l} \in \mathbb{QSCC}(\widetilde{\mathsf{A}}; \mathcal{E})$ and define $l \in \mathbb{P}(\mathsf{A}; \mathcal{E})$ by

$$l_t = \widetilde{l}_t \circ P, \ \ t \geq 0.$$

Then clearly $l_0(a) = \epsilon(P(a))$ for all $a \in \mathsf{A}$, and l is P-invariant. It remains to

check it is a convolution increment process. Choose $t, s \geq 0$ and compute:

$$\begin{aligned} l_{s+t} &= \widetilde{l}_{s+t} \circ P = \left(\widetilde{l}_s \otimes (\sigma_s \circ \widetilde{l}_t)\right) \widetilde{\Delta} P \\ &= \left(\widetilde{l}_s \otimes (\sigma_s \circ \widetilde{l}_t)\right) (P \otimes P)\Delta P = \left(\widetilde{l}_s \otimes (\sigma_s \circ \widetilde{l}_t)\right) (P \otimes P)\Delta \\ &= (l_s \otimes (\sigma_s \circ l_t))\, \Delta. \end{aligned}$$

Conversely, if $l \in \mathbb{P}(\mathsf{A}; \mathcal{E})$ is a P-invariant convolution increment process, with initial condition given by $\epsilon \circ P$, then the process $\widetilde{l} \in \mathbb{P}(\widetilde{\mathsf{A}}; \mathcal{E})$, defined simply by the restriction of l is a QS convolution cocycle on $\widetilde{\mathsf{A}}$ – again the only thing to be checked is the convolution increment property: for all $s, t \geq 0$, $a \in \widetilde{\mathsf{A}}$,

$$\begin{aligned} \widetilde{l}_{s+t}(a) &= l_{s+t}(a) = (l_s \otimes (\sigma_s \circ l_t))\, \Delta(a) = (l_s \otimes (\sigma_s \circ l_t))\, \Delta P(a) \\ &= (l_s \otimes (\sigma_s \circ l_t))\, (P \otimes P)\Delta P(a) = \left(\widetilde{l}_s \otimes (\sigma_s \circ \widetilde{l}_t)\right) \widetilde{\Delta}(a). \end{aligned}$$

□

Remark 4.8.20. If $\epsilon = \epsilon \circ P$ (as is in the case of Delsarte C^*-hyperbialgebras, but usually not for double coset bialgebras), the processes l in the proof of the above theorem are obviously QS convolution cocycles. Assuming this is the case, and noting that Markov-regularity is clearly preserved under the correspondence in the above fact, it is easily checked that if $\varphi \in CB(\widetilde{\mathsf{A}}; B(\widehat{\mathsf{k}}))$ then $\widetilde{l} := l^\varphi \in \mathbb{QSCC}(\widetilde{\mathsf{A}}; \mathcal{E})$ corresponds to the process $l^\psi \in \mathbb{QSCC}(\mathsf{A}; \mathcal{E})$ generated by $\psi := \varphi \circ P \in CB\left(\mathsf{A}; B(\widehat{\mathsf{k}})\right)$. There is an analogous correspondence on the level of weak QS convolution cocycles.

4.9 Towards QS convolution cocycles on locally compact quantum groups

In this short section we discuss a possible approach to QS convolution cocycles on multiplier C^*-bialgebras. No satisfactory results concerning the existence

and characterisation of cocycles are known in this generality, and the whole section should be considered as an announcement of a problem rather than the formulation of solutions. The motivation for considering this question lies in the recently developed theory of locally compact quantum groups.

Multiplier algebras and multiplier C^*-bialgebras

Recall that if A is a C^*-algebra then a closed (twosided, selfadjoint) ideal I of A is called *essential* if for all $a \in \mathsf{A}$ the equality $a\mathsf{I} = \{0\}$ implies $a = 0$.

Definition 4.9.1. Let A be a C^*-algebra. The multiplier algebra of A, denoted by $\mathsf{M}(\mathsf{A})$, is the biggest C^*-algebra containing A as an essential ideal.

The definition above requires certain comments. The biggest is understood in the following sense: whenever B is another C^*-algebra containing A as an essential ideal, the identity map on A extends to an injective *-homomorphism from B to $\mathsf{M}(\mathsf{A})$. Therefore the uniqueness of $\mathsf{M}(\mathsf{A})$ up to an isomorphism follows directly from the definition. The easiest way to show the existence is to exhibit a concrete model for $\mathsf{M}(\mathsf{A})$. Below we describe a model of so-called *double centralizers* (known also as *multipliers*).

Let A be a C^*-algebra. By a double centralizer on A we understand a pair of maps $S, T : \mathsf{A} \to \mathsf{A}$ satisfying the following condition:

$$T(a)b = aS(b), \quad a, b \in \mathsf{A}.$$

It is easy to see that both S and T must be linear module maps ($aT(b) = T(ab)$, $S(a)b = S(ab)$ for all $a, b \in \mathsf{A}$) and the Closed Graph Theorem implies they are continuous. The vector space $\mathsf{DC}(\mathsf{A})$ of all double centralisers equipped with the norm $\|(S,T)\| := \|S\| (= \|T\|)$ is a Banach space. Define the multiplication by $(S_1, T_1) \cdot (S_2, T_2) := (S_2 S_1, T_1 T_2)$ and the adjoint operation by $(S, T)^* = (S', T')$, where $S'(a) = (T(a^*))^*$, $T'(a) = (S(a^*))^*$ for all $a \in \mathsf{A}$ (note that $(S,T)^* = (T^\dagger, S^\dagger)$ in the earlier nomenclature). The norm introduced before is submultiplicative and one may check it satisfies the C^*-condition, so that

$\mathsf{DC}(\mathsf{A})$ is a C^*-algebra. The algebra A may be embedded in $\mathsf{DC}(\mathsf{A})$ via the identification $a \mapsto (L_a, R_a)$ (left and right multiplication by a). It may be shown that the image of A in this embedding is an essential ideal of $\mathsf{DC}(\mathsf{A})$, and that $\mathsf{DC}(\mathsf{A})$ satisfies the maximality condition required in the definition of the multiplier algebra.

Note that the multiplier algebra $\mathsf{M}(\mathsf{A})$ is always unital, and if A is unital then $\mathsf{M}(\mathsf{A}) \cong \mathsf{A}$. If X is a locally compact topological space and $\mathsf{A} = C_0(X)$, then $\mathsf{M}(\mathsf{A}) \cong C_b(X) \cong C(\beta X)$, where $C_0(X)$ denotes the algebra of continuous functions on X vanishing at infinity, $C_b(X)$ the algebra of continuous bounded functions and βX the Stone-Čech compactification of X.

Apart from the norm topology, there is another useful locally convex topology on $\mathsf{M}(\mathsf{A})$, the so-called *strict topology*. It is induced by the family of seminorms $\{p_a : a \in \mathsf{A}\} \cup \{p'_a : a \in \mathsf{A}\}$ defined by $p_a(b) = \|ab\|$ and $p'_a(b) = \|ba\|$ for all $a \in \mathsf{A}$ and $b \in \mathsf{M}(\mathsf{A})$. The unit ball of A is strictly dense in the unit ball of $\mathsf{M}(\mathsf{A})$, and *strict* maps (that is bounded linear maps which are strictly continuous on bounded subsets) from A to $\mathsf{M}(\mathsf{B})$ extend uniquely to strict maps on $\mathsf{M}(\mathsf{A})$. The most useful examples of strict maps are *slice maps* and *nondegenerate* *-homomorphisms. By a slice map is meant a map of the form $\omega \otimes \mathrm{id}_{\mathsf{A}} : \mathsf{A} \otimes \mathsf{A} \to \mathsf{A}$, where $\omega \in \mathsf{A}^*$; a *-homomorphism $\psi : \mathsf{A} \to \mathsf{M}(\mathsf{B})$ (where B is another C^*-algebra) is called nondegenerate if $\psi(\mathsf{A})\mathsf{B}$ is dense in B. Note that nondegenerate *-homomorphisms extend to unital *-homomorphisms from $\mathsf{M}(\mathsf{A})$ to $\mathsf{M}(\mathsf{B})$. Elementary proofs of the well-known facts above may be found for example in the paper [Ku1]; for another presentation of multiplier algebras, geared towards (and using the language of) the theory of Hilbert C^*-modules, see the book of [Lan]. In the latter book it is shown that a completely positive map is strict if and only if the image in this map of any (equivalently, every) approximate unit in A is Cauchy with respect to the strict topology in $\mathsf{M}(\mathsf{B})$.

Definition 4.9.2. By a multiplier C^*-bialgebra is meant a C^*-algebra A together with a nondegenerate *-homomorphism $\Delta : \mathsf{A} \to \mathsf{M}(\mathsf{A} \otimes \mathsf{A})$ and a character $\epsilon : \mathsf{A} \to \mathbb{C}$, satisfying the standard coassociativity and counit identi-

ties:

$$(\Delta \otimes \mathrm{id}_{\mathsf{A}}) \circ \Delta = (\mathrm{id}_{\mathsf{A}} \otimes \Delta) \circ \Delta \quad \text{and} \quad (\epsilon \otimes \mathrm{id}_{\mathsf{A}}) \circ \Delta = (\mathrm{id}_{\mathsf{A}} \otimes \epsilon) \circ \Delta = \mathrm{id}_{\mathsf{A}}.$$

The equalities above have to be understood in the only possible way, that is whenever it is necessary one in fact considers continuous extensions of the strict maps in question (e.g. $\Delta \otimes \mathrm{id}_{\mathsf{A}}$ is treated as a map from $\mathsf{M}(\mathsf{A} \otimes \mathsf{A})$ to $\mathsf{M}(\mathsf{M}(\mathsf{A} \otimes \mathsf{A}) \otimes \mathsf{A}) \hookrightarrow \mathsf{M}(\mathsf{A} \otimes \mathsf{A} \otimes \mathsf{A})$). The necessity of introducing such a definition stems from the fact that already for classical, locally compact but noncompact group the comultiplication may be defined on $C_0(G)$ by the formula (4.8.1), but there is no guarantee that it will take values in $C_0(G \times G)$ - it rather maps into $C_b(G \times G) \cong \mathsf{M}(C_0(G) \otimes C_0(G))$. When a multiplier C^*-bialgebra A is unital, it is a C^*-bialgebra.

QS convolution cocycles on multiplier C^*-bialgebras

Let A be a multiplier C^*-bialgebra. The convolution product $\phi_1 \star \phi_2$ retains its meaning as long as the map $\phi_1 \otimes \phi_2$ has a natural extension to the multiplier algebra $\mathsf{M}(\mathsf{A} \otimes \mathsf{A})$. This is the case for example if both ϕ_1, ϕ_2 are bounded functionals ($\phi_1 \otimes \phi_2$ may be then written as a composition of slice maps). Therefore the notion of convolution semigroups of functionals may be used in the context of multiplier C^*-bialgebras without any changes.

Definition 4.9.3. A process $l \in \mathbb{P}_{\mathrm{wb}}(\mathsf{A}; \mathcal{E})$ is a weak convolution increment process on a multiplier C^*-bialgebra A if it satisfies formula (4.2.3) (for the associated convolution semigroups given by (4.2.4)); it is a weak QS convolution cocycle if in addition $l_0(a) = \epsilon(a) I_{\mathcal{F}}$ for all $a \in \mathsf{A}$.

A satisfactory 'strong' definition of a QS convolution cocycle is at the moment available only for nondegenerate *-homomorphic cocycles.

Definition 4.9.4. A process $l \in \mathbb{P}_{\mathrm{cb}}(\mathsf{A}; \mathcal{E})$ is called a nondegenerate *-homomorphic QS convolution cocycle if the following conditions are satisfied:

(i) for each $t \geq 0$ the map l_t is a *-homomorphism which is nondegenerate as a map from A to $B(\mathcal{F}_{[0,t)})$;

(ii) for all $s, t \geq 0$

$$l_{s+t} = l_s \star (\sigma_s \circ l_t); \tag{4.9.1}$$

(iii) for all $a \in \mathsf{A}$

$$l_0(a) = \epsilon(a) I_{\mathcal{F}}.$$

Observe that the formula (4.9.1) makes sense; the (completely bounded) map $l_s \otimes (\sigma_s \circ l_t) : \mathsf{A} \otimes \mathsf{A} \to B(\mathcal{F}_{[0,s)}) \otimes B(\mathcal{F}_{[s,s+t)}) \subset B(\mathcal{F}_{[0,t)})$ is a nondegenerate *-homomorphism, and so extends in a natural way to a (unital) map from $\mathsf{M}(\mathsf{A} \otimes \mathsf{A})$ to $B(\mathcal{F}_{[0,t)})$.

Remark 4.9.5. If A is unital, nondegeneracy means simply that l must be a unital process; in any case, each nondegenerate *-homomorphic QS convolution cocycle on A has a natural extension to a unital *-homomorphic process on $\mathsf{M}(\mathsf{A})$. This may at first suggest that one may reduce the considerations to QS convolution cocycles on $\mathsf{M}(\mathsf{A})$ – note however that $\mathsf{M}(\mathsf{A})$ need not be a C^*-bialgebra in the sense of Definition 4.1.2. In general the algebra $\mathsf{M}(\mathsf{A}) \otimes \mathsf{M}(\mathsf{A})$ is smaller than $\mathsf{M}(\mathsf{A} \otimes \mathsf{A})$.

By a coalgebraic QS differential equation on A (with coefficient $\varphi \in CB\left(\mathsf{A}; B(\widehat{\mathsf{k}})\right)$) is understood again the equation of the form (4.4.1). There is no problem with understanding the notion of a weak solution: the formula (4.4.2) remains meaningful with the same assumptions on the process as before. The strong solutions are more problematic: recall that the first step of the corresponding definition in Section 4.4 required introducing the map ϕ via the R-map. To do it in the context of multiplier C^*-bialgebras one needs to insert additional assumption on φ: it should be 'stably strict', that is the map $\mathrm{id}_{\mathsf{A}} \otimes \varphi : \mathsf{A} \otimes \mathsf{A} \to \mathsf{A} \otimes B(\widehat{\mathsf{k}})$ should extend in a natural way to a map from $\mathsf{M}(\mathsf{A} \otimes \mathsf{A})$ (in general expected to have values in $\mathsf{M}(\mathsf{A} \otimes B(\widehat{\mathsf{k}}))$. Note that although nondegenerate *-homomorphisms are 'stably strict', it is not sufficient

for our purposes. One would like at least to be able to work with the maps φ satisfying the structure relations (4.6.5). The next step, a definition of a process K which is to be actually QS integrated (see (4.4.4)) causes further domain-related problems; despite certain efforts we were not able to identify any natural sufficient conditions on φ allowing for completing this procedure.

As at the moment there is no satisfactory definition of strong solutions of coalgebraic QS differential equation on multiplier C^*-bialgebras, there are also no results on existence of solutions of such equations (uniqueness of weak solutions can be dealt with via methods similar to those of Section 4.4). One of the basic questions would be: what are the minimal assumptions on the coefficient $\varphi : \mathsf{A} \to B(\widehat{\mathsf{k}})$ so that some form of Picard iteration, as in Sections 4.3 and 4.4 is possible? The main problem, as should be apparent from the above considerations, lies in an insufficient understanding of the interplay between operator-space theoretic notions and the theory of multiplier C^*-algebras.

Appendix A

(π_1, π_2)-derivations

In this appendix we give an extension of the innerness theorem of E. Christensen, for completely bounded derivations on a C^*-algebra, to (π_1, π_2)-derivations, and prove automatic complete boundedness for (π, χ)-derivations, when χ is a character.

Definition A.1. Let A be a C^*-algebra and X be a Banach A-bimodule. A map $\delta : \mathsf{A} \to \mathsf{X}$ is called a derivation if for all $a, b \in \mathsf{A}$

$$\delta(ab) = \delta(a)b + a\delta(b).$$

A derivation δ is inner if there exists an element $x \in \mathsf{X}$ such that for all $a \in \mathsf{A}$

$$\delta(a) = ax - xa.$$

The following theorem is a modification (due to J.R. Ringrose) of the celebrated Sakai theorem on boundedness of C^*-algebra derivations.

Theorem A.2 ([Sak], [Rin]). *Let A be a C^*-algebra and X be a Banach A-bimodule. Every derivation $\delta : \mathsf{A} \to \mathsf{X}$ is bounded.*

For certain A-bimodules inner derivations are characterised by their complete boundedness. For a simple proof of the next theorem, and connections

with not necessarily real homomorphisms between C^*-algebras we refer to [Pis$_1$].

Theorem A.3 ([Chr]). *Let $\mathsf{A} \subset B(\mathsf{h})$ be a C^*-algebra, and let $\delta : \mathsf{A} \to B(\mathsf{h})$ be a derivation. Then δ is completely bounded if and only if it is inner.*

For the purpose of this work we are interested in the particular class of Banach A-bimodule-valued derivations captured by the following definition.

Definition A.4. Let A be a C^*-algebra with representations (π_1, h_1), (π_2, h_2). A map $\delta : \mathsf{A} \to B(\mathsf{h}_2; \mathsf{h}_1)$ is called a (π_1, π_2)-derivation if for all $a, b \in \mathsf{A}$

$$\delta(ab) = \pi_1(a)\delta(b) + \delta(a)\pi_2(b).$$

A (π_1, π_2)-derivation δ is inner if it is implemented by an operator $T \in B(\mathsf{h}_2; \mathsf{h}_1)$, in the sense that for all $a \in \mathsf{A}$

$$\delta(a) = \pi_1(a)T - T\pi_2(a).$$

Theorem A.5. *Let A be a C^*-algebra with representations (π_1, h_1), (π_2, h_2) and let $\delta : \mathsf{A} \to B(\mathsf{h}_1; \mathsf{h}_2)$ be a (π_1, π_2)-derivation. Then δ is completely bounded if and only if it is inner.*

Proof. Let (ρ, h) be a faithful representation of A and set $\mathsf{H} = \mathsf{h}_2 \oplus \mathsf{h}_1 \oplus \mathsf{h}$, $\widetilde{\mathsf{A}} = (\pi_2 \oplus \pi_1 \oplus \rho)(\mathsf{A})$. Then $\widetilde{\mathsf{A}}$ is a C^*-subalgebra of $B(\mathsf{H})$, and a map $\widetilde{\delta} : \mathsf{A} \to B(\mathsf{H})$ defined by $(a \in \mathsf{A})$

$$\widetilde{\delta}\left(\begin{pmatrix} \pi_2(a) & & \\ & \pi_1(a) & \\ & & \rho(a) \end{pmatrix}\right) = \begin{pmatrix} 0 & & \\ \delta(a) & 0 & \\ & & 0 \end{pmatrix}.$$

For any $a \in \mathsf{A}$ write $\widetilde{a}$ for a relevant element of $\widetilde{\mathsf{A}}$. Then

$$\begin{aligned}
\widetilde{\delta}(\widetilde{a})\widetilde{b} + \widetilde{a}\widetilde{\delta}(\widetilde{b}) &= \begin{pmatrix} 0 & & \\ \delta(a) & 0 & \\ & & 0 \end{pmatrix} \begin{pmatrix} \pi_2(b) & & \\ & \pi_1(b) & \\ & & \rho(b) \end{pmatrix} \\
&+ \begin{pmatrix} \pi_2(a) & & \\ & \pi_1(a) & \\ & & \rho(a) \end{pmatrix} \begin{pmatrix} 0 & & \\ \delta(b) & 0 & \\ & & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & & \\ \delta(a)\pi_2(b) + \pi_1(a)\delta(b) & 0 & \\ & & 0 \end{pmatrix} = \begin{pmatrix} 0 & & \\ \delta(ab) & 0 & \\ & & 0 \end{pmatrix} \\
&= \widetilde{\delta}(\widetilde{ab}) = \widetilde{\delta}(\widetilde{a}\widetilde{b}),
\end{aligned}$$

so $\widetilde{\delta}$ is a derivation. Observe that δ is inner if and only if $\widetilde{\delta}$ is; if $\widetilde{\delta}$ is implemented by $T \in B(\mathsf{H})$, δ is implemented by $P_{\mathsf{h}_1} T P_{\mathsf{h}_2}$ (a (2,1) 'matrix coefficient' of T). The claim now follows from Theorem A.3. $\square$

Theorem A.6. *Let A be a C^*-algebra with a representation (π, h) and a character (nonzero multiplicative functional) χ. Then every (π, χ)-derivation is completely bounded.*

Proof. Let $\delta : \mathsf{A} \to B(\mathbb{C}; \mathsf{h}) = |\mathsf{h}\rangle$ be a (π, χ)-derivation. By Theorem A.2 δ is bounded. Without loss of generality we may suppose that the C^*-algebra A and representation π are both unital; if necessary by extending π, χ and δ to the unitisation of A in the natural way:

$$(a, z) \mapsto \pi(a) + zI_{\mathsf{h}}, \quad (a, z) \mapsto \chi(a) + z \quad \text{and} \quad (a, z) \mapsto \delta(a).$$

Denote by K a kernel of χ, and let $P : \mathsf{A} \to K$ denote the canonical (bounded) projection onto K:

$$P(a) = a - \chi(a)1, \quad a \in \mathsf{A}.$$

Then K is a maximal ideal, and as such also a (nonunital) C^* -subalgebra of A. Let $\{u_i : i \in I\}$ be an approximate unit of K (contained in the unit ball).

Its existence guarantees that the set $K_2 = \mathrm{lin}\{ab : a, b \in K\}$ is dense in K. For any $n \in \mathbb{N}$, $a_1, \ldots, a_n, b_1, \ldots, b_n \in K$

$$\sum_{k=1}^{n} \delta(a_k^*)^* \delta(b_k) = \lim_{i \in I} \left(\sum_{k=1}^{n} \delta(a_k^* u_i)^* \delta(b_k u_i) \right) = \lim_{i \in I} \left(\sum_{k=1}^{n} (\pi(a_k^*)\delta(u_i))^* \pi(b_k)\delta(u_i) \right)$$

$$= \lim_{i \in I} \delta(u_i)^* \pi(\sum_{k=1}^{n} a_k b_k) \delta(u_i),$$

so

$$\left| \sum_{k=1}^{n} \delta(a_k^*)^* \delta(b_k) \right| \leq \|\delta\|^2 \| \sum_{k=1}^{n} a_k b_k \|.$$

This allows us to define a bounded functional $\widetilde{\gamma} : K_2 \to \mathbb{C}$ by

$$\widetilde{\gamma}(\sum_{k=1}^{n} a_k b_k) = \sum_{k=1}^{n} \delta(a_k^*)^* \delta(b_k)$$

(its continuous extension to K will be extended by the same letter). Put then

$$\gamma = \widetilde{\gamma} \circ P.$$

For any $a, b \in \mathsf{A}$

$$\delta(a)^* \delta(b) = \delta(a - \chi(a)1)^* \delta(b - \chi(b)1) = \widetilde{\gamma}((a^* - \chi(a^*)1)(b - \chi(b)1))$$

$$= \gamma((a^* - \chi(a^*)1)(b - \chi(b)1)) = \gamma(a^* b) - \gamma(a^*)\chi(b) - \chi(a^*)\gamma(b), \tag{A.1}$$

where we used the fact that $\delta(1) = 0$. The last formula has a natural matricial equivalent. Fix $n \in \mathbb{N}$ and let $\delta_n : M_n(\mathsf{A}) \to M_n(B(\mathbb{C}; H)) = B(\mathbb{C}^n; H^n)$, $\gamma_n : M_n(\mathsf{A}) \to M_n(\mathbb{C})$ and $\chi_n : M_n(\mathsf{A}) \to M_n(\mathbb{C})$ be respective tensorizations of δ, γ and χ. Then for any $\widetilde{a} = [a_{i,j}] \in M_n(A)$, $\widetilde{b} = [b_{i,j}] \in M_n(A)$

$$\delta_n(\widetilde{a})^* \delta_n(\widetilde{b}) = \gamma_n(\widetilde{a}^* \widetilde{b}) + \gamma_n(\widetilde{a}^*)\chi_n(\widetilde{b}) + \chi_n(\widetilde{a}^*)\gamma_n(\widetilde{b}) \tag{A.2}$$

Indeed, for any $i, j \in \{1, \ldots, n\}$ equation (A.1) implies

$$\begin{aligned}(\delta_n(\widetilde{a})^* \delta_n(\widetilde{b}))_{i,j} &= \sum_{k=1}^{n} \delta(a_{k,i})^* \delta(b_{k,j}) = \\ &\sum_{k=1}^{n} \left(\gamma(a_{k,i}^* b_{k,j}) - \gamma(a_{k,i}^*) \chi(b_{k,j}) - \chi(a_{k,i}^*) \gamma(b_{k,j}) \right).\end{aligned}$$

Complete boundedness of δ follows easily from the estimate:

$$\|\delta_n\|^2 \leq \|\gamma_n\| + 2\|\gamma_n\| \|\chi_n\| = 3\|\gamma\|$$

and arbitrariness of $n \in \mathbb{N}$. □

Theorems A.5 and A.6 yield the following corollaries:

Corollary A.7. *Let* A *be a* C^**-algebra with with a representation* (π, h) *and a character* χ*. Then every* (π, χ)*-derivation is inner.*

Corollary A.8. *If* A *is a* C^**-algebra with a character* χ *then every* (χ, χ)*-derivation on* A *vanishes.*

Appendix B

Hopf *-algebras and their corepresentations

In this appendix we present the basics of the theory of Hopf *-algebras and their corepresentations, following the presentation in [KlS]. In the second part we give the relevant definitions for compact quantum groups.

Definition B.1. A unital bialgebra $\mathcal{A}$ (with multiplication m) is called a Hopf algebra if there exists a linear mapping $\mathcal{S} : \mathcal{A} \to \mathcal{A}$, called the antipode or the coinverse of $\mathcal{A}$, such that for all $a \in \mathcal{A}$

$$m(\mathcal{S} \otimes \mathrm{id}_{\mathcal{A}})\Delta(a) = m(\mathrm{id}_{\mathcal{A}} \otimes \mathcal{S})\Delta(a) = \epsilon(a)1. \tag{B.1}$$

Note that the antipode may be viewed as a convolution inverse of the identity mapping; in the convolution notation (B.1) takes form

$$(\mathcal{S} \star \mathrm{id}_{\mathcal{A}})(a) = (\mathrm{id}_{\mathcal{A}} \star \mathcal{S})(a) = \epsilon(a), \quad a \in \mathcal{A}.$$

Proposition B.2. *The antipode of a Hopf algebra is necessarily a unital antihomomorphism and a counital anti-coalgebra morphism, that is*

$$\mathcal{S}(1) = 1, \quad \mathcal{S}(ab) = \mathcal{S}(b)\mathcal{S}(a), \quad a, b \in \mathcal{A},$$

and

$$\epsilon \circ \mathcal{S} = \epsilon, \quad \Delta \circ \mathcal{S} = \tau \circ (\mathcal{S} \otimes \mathcal{S}) \circ \Delta,$$

where τ denotes the tensor flip on $\mathcal{A} \odot \mathcal{A}$.

Definition B.3. A unital *-bialgebra is called a Hopf *-algebra if it is a Hopf algebra.

Although the definition of a Hopf *-algebra does not specify the behaviour of the antipode under the involution, it follows from the definitions that for all $a \in \mathcal{A}$

$$\mathcal{S}(\mathcal{S}(a^*)^*) = a.$$

In particular, the antipode of a Hopf *-algebra is always invertible.

Definition B.4. Let $\mathcal{C}$ be a coalgebra. A corepresentation of $\mathcal{C}$ on a vector space V is a linear map $\varphi : V \to V \odot \mathcal{A}$ such that

$$(\mathrm{id}_V \otimes \Delta) \circ \varphi = (\varphi \otimes \mathrm{id}_{\mathcal{A}}) \circ \varphi, \quad (\mathrm{id}_V \otimes \epsilon) \circ \varphi = \mathrm{id}_V.$$

A linear subspace $W \subset V$ is called φ-invariant if $\varphi(W) \subset W \odot \mathcal{A}$. Then the restriction of φ to W is also a corepresentation of $\mathcal{C}$, called a subcorepresentation of φ. A corepresentation φ is called irreducible if it does not have any nontrivial (that is different from itself and from zero) subcorepresentations.

It is obvious that the coproduct of $\mathcal{C}$ is itself a corepresentation of $\mathcal{C}$ (on $\mathcal{C}$). The following fact may therefore be viewed as a generalisation of the Fundamental Theorem on Coalgebras:

Fact B.5. *If φ is a corepresentation of a coalgebra $\mathcal{C}$ on a vector space V, then any element of V is contained in a finite-dimensional φ-invariant subspace.*

Assume now that V is a finite-dimensional Hilbert space and let $(e_1, \ldots, e_n)$ be an orthonormal basis of V. Whenever φ is a representation of a Hopf *-algebra $\mathcal{A}$ on V, there exist uniquely determined elements $v_{ij} \in \mathcal{A}$

$(i, j = 1, \ldots, n)$, called the *matrix coefficients of φ with respect to the basis* $(e_1, \ldots, e_n)$, such that for each $j = 1, \ldots, n$

$$\varphi(e_j) = \sum_{i=1}^{n} e_i \otimes v_{ij}.$$

Definition B.6. A corepresentation of a Hopf *-algebra $\mathcal{A}$ on a finite-dimensional Hilbert space V is called unitary if for some (equivalently, for any) orthonormal basis of V the matrix coefficients of φ with respect to this basis satisfy the conditions:

$$\mathcal{S}(v_{ij}) = (v_{ji})^*, \quad i, j = 1, \ldots, n. \tag{B.2}$$

The conditions (B.2) are equivalent to the equality $v^*v = vv^* = I_{M_n(\mathcal{A})}$, where $v = (v_{ij})_{i,j=1}^n \in M_n(\mathcal{A})$. This explains the motivation behind the above definition.

Recall the definition of a compact quantum group (Definition 4.8.6). The notion of unitary corepresentations in the context of compact quantum groups needs to be slightly different from the one given below, as in principle we do not want to exclude infinite-dimensional corepresentations. Let $K(\mathsf{H})$ denote the C^*-algebra of all compact operators on a Hilbert space H and recall that for any C^*-algebra B its multiplier algebra (see Section 4.9) is denoted by $\mathsf{M}(\mathsf{B})$.

Definition B.7. Let (A, Δ) be a compact quantum group, and H be a Hilbert space. By a unitary corepresentation of A acting on H is understood a unitary $U \in \mathsf{M}(K(\mathsf{H}) \otimes \mathsf{A})$ satisfying the condition

$$(\mathrm{id}_{\mathsf{H}} \otimes \Delta)U = U_{12}U_{13}$$

(in the above formula operators $U_{12}, U_{13} \in \mathsf{M}(K(\mathsf{H}) \otimes \mathsf{A} \otimes \mathsf{A})$ are defined via the so-called leg notation, and certain natural extensions of maps in question are implicitly understood). A corepresentation U is called irreducible if for any projection $P \in B(\mathsf{H})$ the commutation relation $(P \otimes 1_{\mathsf{A}})U = U(P \otimes 1_{\mathsf{A}})$ implies $P = 0$ or $P = 1_{B(\mathsf{H})}$.

S. L. Woronowicz proved that every irreducible corepresentation of a compact quantum group must be finite-dimensional (that is, H is finite-dimensional). Note that in such a case $K(\mathsf{H}) = B(\mathsf{H})$ is unital, there is no need to consider multiplier algebras and unitaries U_{12} and U_{13} in $B(\mathsf{H}) \otimes \mathsf{A} \otimes \mathsf{A}$ are defined simply by

$$U_{12} = U \otimes 1_{\mathsf{A}}, \quad U_{13} = (\mathrm{id}_{\mathsf{H}} \otimes \tau)(U \otimes 1_{\mathsf{A}})(\mathrm{id}_{\mathsf{H}} \otimes \tau),$$

where $\tau : \mathsf{A} \otimes \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ is the tensor flip.

To complete the list of notions used in Section 4.8 one more definition is needed.

Definition B.8. Let U be a finite-dimensional unitary corepresentation of a compact quantum group A acting on a Hilbert space H. By matrix coefficients of U with respect to an orthonormal basis $(e_1, \dots, e_n)$ of H are understood elements $v_{ij} \in \mathsf{A}$ defined by

$$v_{i,j} = (\langle e_i| \otimes 1_{\mathsf{A}})U(|e_j\rangle \otimes 1_{\mathsf{A}}), \quad i, j = 1, \dots n.$$

One may check that the notions of corepresentations, their irreducibility and matrix coefficients introduced above are consistent for finite-dimensional compact quantum groups (which are also Hopf *-algebras). For more information on the vast topic of topological quantum groups we refer to [Ku2] and references therein.

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